Logistic Differential Equations

Populations increase exponentially, but there has to be a limit (there can only be so many people on Gilligan's Island).

We need to modify our initial function:

$$P=P_o e^{kt}$$

in order to compensate for these real world restrictions.

The Logistic Model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

where
k = growth constant

K = carrying capacity (the maximum)

P = Population

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity

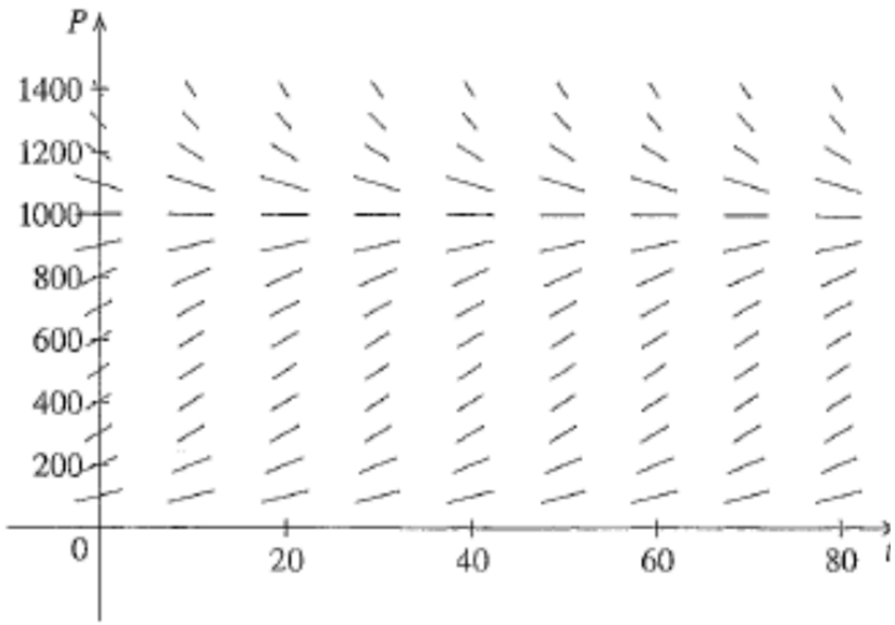
Notice that if P is much smaller than K, then the rate is more affected by the Population (P/K -> 0)

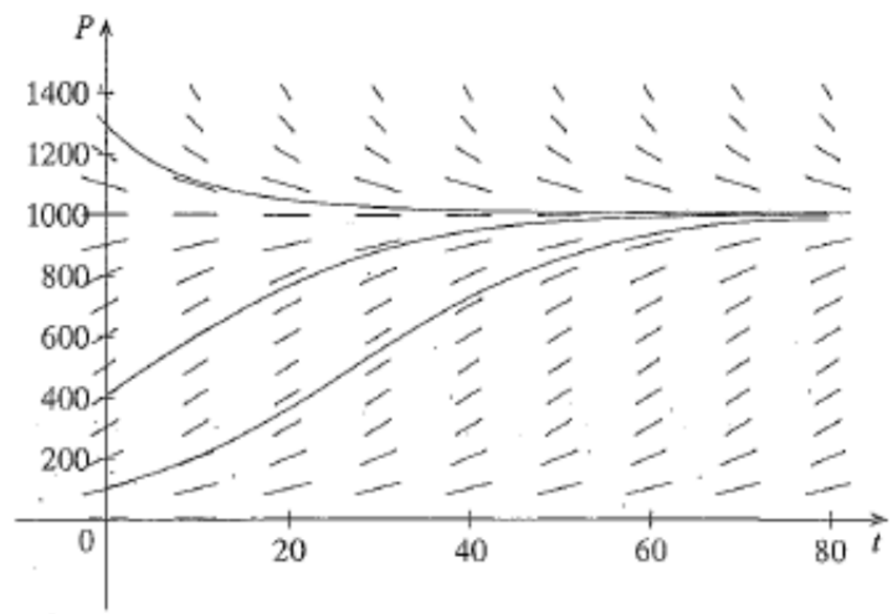
If P is closer to K, then the quantity (1-P/K) approaches 0, and thus the rate approaches 0.

Here is a slope field of the Differential Equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

where k=0.08, K=1000. What do we notice now?





If $\frac{dP}{dt} = P\left(4 - \frac{P}{6000}\right)$ what is the carrying capacity?

Turn the equation into the form $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$

$$\frac{dP}{dt} = P\left(4 - \frac{P}{6000}\right)$$
$$\frac{dP}{dt} = 4P\left(1 - \frac{P}{24000}\right)$$

So carrying capacity, K, is 24,000

If $\frac{dP}{dt} = P\left(4 - \frac{P}{6000}\right)$

at what population will there be the fastest growth?

We're looking for when the derivative (the equation we're given) is at a maximum. This occurs when the second derivative is zero.

$$P' = P\left(4 - \frac{P}{6000}\right)$$

$$P' = 4P - \frac{1}{6000}P^2$$

$$P'' = 4 - \frac{1}{3000}P$$

(yes, we're actually taking the derivative with respect to P - not t. We only need to know at what population the greatest rate of change exists. Finding the time this occurs is beyond the scope of the class).

$$0 = 4 - \frac{1}{3000}P$$

$$-4(-3000) = P$$

$$12000 = P$$

This shows us that the population is growing fastest when P = 12,000

In these problems, the fastest growth occurs at 0.5K (half the carrying capacity).

Check out the graph of this situation. Notice:
carrying capacity (K) = 24,000
Inflection point (maximum derivative) = 12,000
* half of the K, right?

Initial population = 800