8.3 Using Vectors with Parametric Functions

$x'(t) = \frac{dx}{dt}$ is the rate at which the x-coordinate is changing with respect to t or the velocity of the particle in the horizontal direction.

$y'(t) = \frac{dy}{dt}$ is the rate at which the y-coordinate is changing with respect to t or the velocity of the particle in the vertical direction.

$\left\langle x(t), y(t) \right\rangle$ is the **position vector** at any time *t*.

$\left\langle x'(t), y'(t) \right\rangle$ is the **velocity vector** at any time *t*.

$\left\langle x''(t), y''(t) \right\rangle$ is the **acceleration vector** at any time *t*.

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the **speed of the particle** or the **magnitude (length) of the velocity vector**.

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the **length of the arc (or arc length) of the curve from** $t = a$ to $t = b$ or the **distance traveled by the particle from** $t = a$ to $t = b$.

Example 1 (no calculator):

A particle moves in the *xy*-plane so that at any time *t*, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

(a) Find the velocity vector when $t = 1$.

Solution:

$$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{d}{dt}\left(t^3 + 4t^2\right), \frac{d}{dt}\left(t^4 - t^3\right) \right\rangle = \left\langle 3t^2 + 8t, 4t^3 - 3t^2 \right\rangle$$

$$v(1) = \langle 11, 1 \rangle$$

(b) Find the acceleration vector when $t = 2$.

Solution:

$$a(t) = \left\langle \frac{d}{dt}\left(\frac{dx}{dt}\right), \frac{d}{dt}\left(\frac{dy}{dt}\right) \right\rangle = \left\langle \frac{d}{dt}\left(3t^2 + 8t\right), \frac{d}{dt}\left(4t^3 - 3t^2\right) \right\rangle = \left\langle 6t + 8, 12t^2 - 6t \right\rangle$$

$$a(2) = \langle 20, 36 \rangle$$

Example 2 (no calculator):

A particle moves in the *xy*-plane so that at any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^2 + 3t$, $y(t) = t^3 - 3t^2$. Find the magnitude of the velocity vector when $t = 1$.

For our problem, $\frac{dx}{dt} = \frac{d}{dt}\left[t^2 + 3t\right] = 2t + 3$ and $\frac{dy}{dt} = \frac{d}{dt}\left[t^3 - 3t^2\right] = 3t^2 - 6t$.

$$\text{Magnitude of velocity vector} = \left.\sqrt{(2t+3)^2 + (3t^2 - 6t)^2}\,\right|_{t=1} = \sqrt{25+9} = \sqrt{34}.$$

A particle moves in the *xy*-plane so that
$x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \le t \le 2\pi$.

The path of the particle intersects the *x*-axis twice. Write an expression that represents the distance traveled by the particle between the two *x*-intercepts. Do not evaluate.

The path of the particle intersects the *x*-axis at the points where the *y*-component is equal to zero. Note that $1 - 2\sin t = 0$ when $\sin t = \frac{1}{2}$. For $0 \le t \le 2\pi$, this will occur when $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. Since $\frac{dx}{dt} = \frac{d}{dt}\left[\sqrt{3} - 4\cos t\right] = 4\sin t$ and $\frac{dy}{dt} = \frac{d}{dt}\left[1 - 2\sin t\right] = -2\cos t$, the distance traveled by the particle is Distance = $\int_{\pi/6}^{5\pi/6} \sqrt{(4\sin t)^2 + (-2\cos t)^2}\, dt$.