Calculus with Parametric Curves

If a smooth curve C given by the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

the slope of C at the point (x, y) is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{where} \quad \frac{dx}{dt} \neq 0$$

The second derivative for a point (x, y) on curve C is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \boxed{\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}}$$

Arc Length of Parametric Curves

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int \sqrt{(dx^2 + dy^2)}$$

$$\sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx$$

Example 1

Given the parametric equations $x = 2\sqrt{t}$ and $y = 3t^2 - 2t$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$2t^{1/2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t-2}{t^{-1/2}} = (6t-2)t^{1/2} = 6t^{3/2} - 2t^{1/2}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{(dx)^2} = \frac{9t^{1/2} - t^{-1/2}}{t^{-1/2}} = (9t^{1/2} - t^{-1/2})t^{1/2} = 9t - 1$$

Example 2

$y - y_1 = m(x - x_1)$

Given the parametric equations $x = 4\cos t$ and $y = 3\sin t$ write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{-4\sin t} = -\frac{3}{4}\cot t$$

$$= -\frac{3}{4}\cot\left(\frac{3\pi}{4}\right) = -\frac{3}{4}(-1) = \boxed{\frac{3}{4} = m}$$

$$x\left(\frac{3\pi}{4}\right) = 4\cos\frac{3\pi}{4} = 4\cdot\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-2\sqrt{2}}$$

$$y\left(\frac{3\pi}{4}\right) = 3\sin\left(\frac{3\pi}{4}\right) = \boxed{3\cdot\frac{\sqrt{2}}{2}}$$

$$\boxed{y - \frac{3\sqrt{2}}{2} = \frac{3}{4}\left(x + 2\sqrt{2}\right)}$$

Example 3

Find all points of horizontal and vertical tangency to the curve defined by the parametric equations

$$x = t^2 + t \quad \text{and} \quad y = t^2 - 3t + 5$$

$\frac{dy}{dx} = \frac{2t-3}{2t+1}$

HOR TANG. WHEN $2t-3=0$

$+3 \quad +3$

$\frac{2t}{2} = \frac{3}{2}$

$t = 3/2$

$x\left(\frac{3}{2}\right) = \frac{3}{2}^2 + \frac{3}{2}$

$= \frac{9}{4} + \frac{6}{4}$

$= \frac{15}{4}$

$y\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 5$

$= \frac{9}{4} - \frac{18}{4} + \frac{20}{4}$

$= \frac{11}{4}$

$\left(\frac{15}{4}, \frac{11}{4}\right)$

VER TANG:

$2t+1=0$

$t = -\frac{1}{2}$

$\left(-\frac{1}{4}, \frac{27}{4}\right)$

Example 4

Set up an integral expression for the arc length of the curve given by the parametric equations

$$x = t^2 + 1 \quad \text{and} \quad y = 4t^3 - 1 \qquad 0 \le t \le 1$$