Integrating Rational Functions

Sometime we need long division:

$$\int \frac{x^2+2x-9}{4x+5}\,dx$$

Long division:

$$\begin{array}{r} \frac{1}{4}x + \frac{3}{16} + \dfrac{\frac{-159}{16}}{4x+5} \\ 4x+5 \overline{\big)\, x^2 + 2x - 9} \\ -\left(x^2 + \frac{5}{4}x\right) \downarrow \\ \hline \frac{3}{4}x - 9 \\ -\left(\frac{3}{4}x + \frac{15}{16}\right) \\ \hline -9\frac{15}{16} = \frac{-159}{16} \end{array}$$

$$= \int \left(\frac{1}{4}x + \frac{3}{16} + \frac{\frac{-159}{16}}{4x+5} \right) dx$$

$$= \frac{1}{4}\int x\,dx + \frac{3}{16}\int dx - \frac{159}{16}\int \frac{dx}{4x+5}$$

$$\frac{1}{4}\frac{x^2}{2} + \frac{3}{16}x - \frac{159}{16}\ln|4x+5| \cdot \frac{1}{4} + C$$

Other times we need to use
"Partial Fraction Decomposition"

Use this is the function is not improper.

$$\int \frac{2x^2 - 3x - 1}{x^3 - x}\, dx$$

Partial Fraction Decomposition

$$\frac{2x^2-3x-1}{x^3-x}=\frac{2x^2-3x-1}{x(x+1)(x-1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$$

$$2x^2-3x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$$

$$=Ax^2-A+Bx^2-Bx+Cx^2+Cx$$

$$=(A+B+C)x^2+(-B+C)x-A$$

$$A+B+C=2 \qquad -B+C=-3 \qquad -A=-1$$

$$A=1,\ B=2,\ C=-1$$

$$\frac{1}{x}+\frac{2}{x+1}+\frac{-1}{x-1}$$

So now we can rewrite the integral:

$$\int \frac{2x^2-3x-1}{x^3-x}\,dx = \int\left[\frac{1}{x} + \frac{2}{x+1} + \frac{-1}{x-1}\right]dx$$

$$= \ln|x| + 2\ln|x+1| - \ln|x-1| + c$$

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. *Divide if improper:* If $N(x)/D(x)$ is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 (below) to the proper rational expression $N_1(x)/D(x)$.

2. *Factor denominator:* Completely factor the denominator into factors of the form

$$(px + q)^m \qquad \text{and} \qquad (ax^2 + bx + c)^n$$

3. *Linear factors:* For *each* factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdot \cdot \cdot + \frac{A_m}{(px + q)^m}$$

4. *Quadratic factors:* For *each* factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdot \cdot \cdot + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Try this one:

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$$

$$\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\frac{x^2-x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$