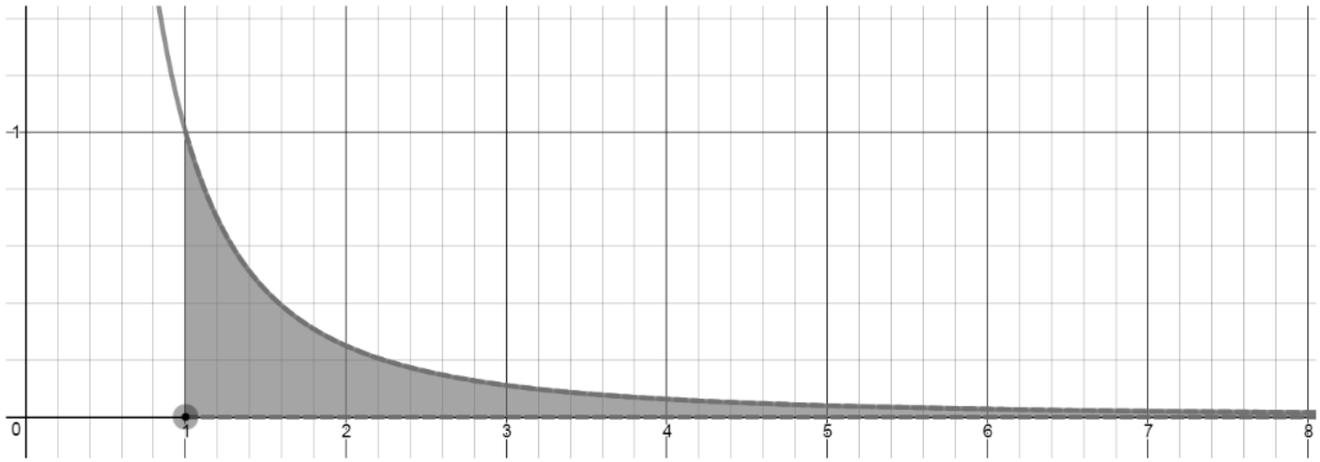
Let's check out the area under the curve $f(x) = \frac{1}{x^2}$ on the interval (1, ∞).

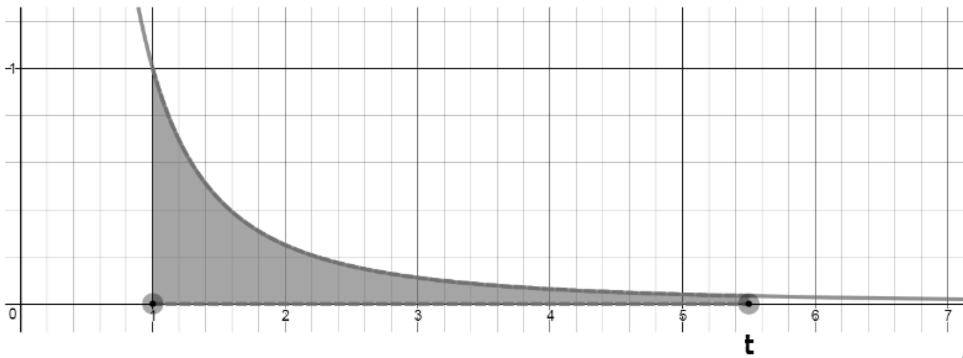
$$f(x) = \int_1^{\infty} \frac{1}{x^2}\,dx$$



The graph makes it look like the area might be ∞, but we would want to evaluate the integral to make sure. The problem is, we can't evaluate F(∞) - F(1) since F(∞) is undefined; ∞ isn't really a number.

So we'll try to evaluate the integral using limits.

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$$



$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t} - -\frac{1}{1}\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t}\right) - \lim_{t\to\infty}(-1)$$

$$= 0 - -1 = \boxed{1}$$

As we move t farther to the left, the limit of the area under the function approaches 1.

We can use this method to solve other improper integrals.

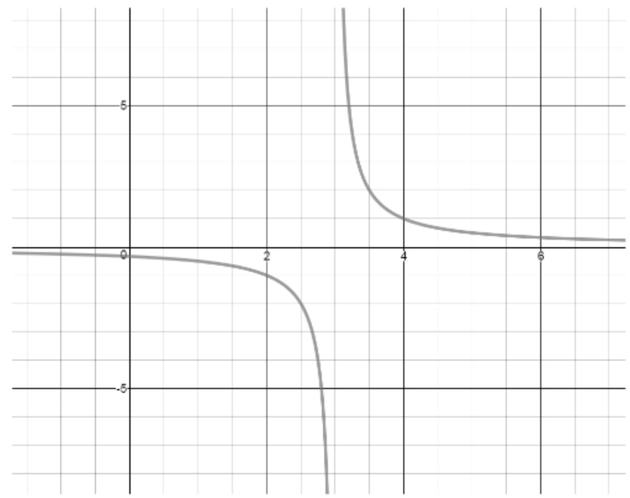
A quick study of what's an improper integral

$$y = \int_{-\infty}^{6} 7x^2\,dx$$

Anything with ∞ as a bound

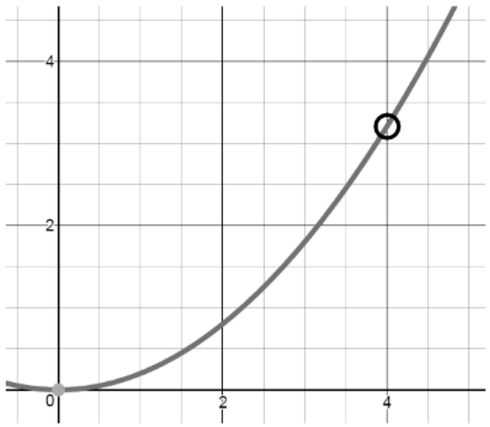
$$y = \int_{2}^{5} \frac{1}{(x-3)}\,dx$$

Anytime we're integrating over an <u>infinite</u> discontinuity (here there's a vertical asymptote at x = 3).



These might look like improper integrals, but actually the laws of integration allow us to evaluate the answers.

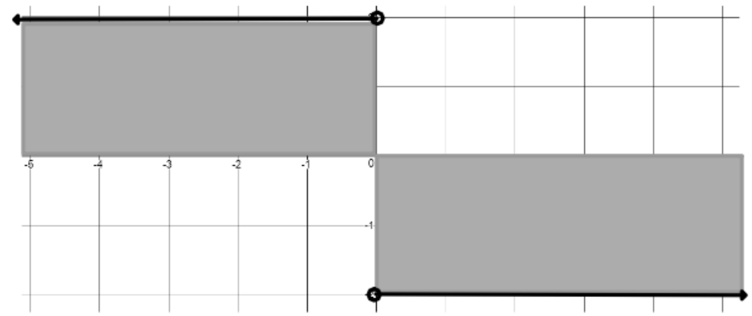
$$y = \int_1^6 \frac{0.2x^2(x-4)}{x-4}\,dx$$



This example is NOT improper, since it's ok to integrate over a <u>removable</u> discontinuity (just "remove" the hole and integrate).

$$y = \int_{-5}^{5} f(x)\,dx$$

$$\text{where } f(x) = 2 \quad \text{when } x < 0$$
$$\text{and } f(x) = -2 \quad \text{when } x > 0$$



This can also be integrated, since it's OK to integrate across a <u>jump</u> discontinuity (just break it up into piece-wise functions).

And of course, any integrand that is continuous makes for a proper integral.

Let's try out our method of taking limits to evaluate an improper integral.

$$f(x) = \int_2^5 \frac{1}{(x-3)}\,dx$$

Split it up: $\lim_{b\to 3^-}\int_2^b \frac{1}{(x-3)}\,dx + \lim_{b\to 3^+}\int_b^5 \frac{1}{(x-3)}\,dx$ ← Use left and right hand limits when dealing with infinite discontinuities.

$$\lim_{b\to 3^-}\ln|x-3|\,\Big|_2^b + \lim_{b\to 3^+}\ln|x-3|\,\Big|_b^5$$

$$\lim_{b\to 3^-}(\ln(b-3)-\ln 1) + \lim_{b\to 3^+}(\ln 2 - \ln(b-3))$$

STOP! Even though it looks as if they'd cancel, $\lim_{b\to 3^-}\ln(b-3)$ does not exist.

This integral is <u>divergent</u> - the limit does not exist!

Let's try one that is convergent (there is an answer):

$$\int_{-\infty}^{1} -e^{x}\,dx$$

$$-\lim_{b\to -\infty}\int_{b}^{1} e^{x}\,dx$$ Factor out that negative, create the limit

$$-\lim_{b\to -\infty} e^{x}\Big|_{b}^{1}$$

$$-\lim_{b\to -\infty}\left(e^{1}-e^{b}\right)$$

$$-\left(\lim_{b\to -\infty} e-\lim_{b\to -\infty} e^{b}\right)$$ To figure out $\lim_{b\to -\infty} e^{b}$ think about the graph of e^{x}

$$-(e-0) = -e$$

An improper integral is **convergent** if its corresponding limit exists (there's an answer).

An improper integral is **divergent** if its corresponding limit does not exist (the area is some form of infinity).

Assuming a continuous graph on the interval,

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Must be convergent
You can choose, but usually go with a=0

For Infinite Discontinuities

$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$	Use on [a, b), where f(x) is discontinuous at b
$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$	Use on (a, b], where f(x) is discontinuous at a
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$	Use if f(x) has a discontinuity at c, and a<c<b

Must be convergent

Try this:

$$\int_1^{\infty} \frac{1}{x}\, dx$$

$$\lim_{t \to \infty} \int_1^{t} \frac{1}{x}\, dx$$

$$\lim_{t \to \infty} \left(\ln x \,\Big|_1^{t} \right)$$

$$\lim_{t \to \infty} (\ln t - \ln 1)$$

Since $\lim_{t \to \infty} (\ln t) = \infty$

this function is divergent.

Kind of weird, since earlier we found that $\int_1^{\infty} \frac{1}{x^2}\, dx$ to be convergent.

Here's a slick little rule:

$$\int_1^\infty \frac{1}{x^p}\,dx$$

is convergent if $p > 1$
is divergent if $p \leq 1$