7.1 Integration by Parts

As we learned once upon a time, the Substitution Method "undoes" the Chain Rule.

Here we learn a method to undo the Product Rule, which is called

Integration by Parts. Look out!

Let's start with the boring old product rule:

Just watch: $\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right]=f\left(x\right)g'\left(x\right)+f'\left(x\right)g\left(x\right)$

What is happening in the step below? FTC pt 1

$$f\left(x\right)g\left(x\right)=\int[f(x)g'(x)+f'(x)g(x)]\,dx$$

$$f\left(x\right)g\left(x\right)=\int f(x)g'(x)\,dx+\int f'(x)g(x)\,dx$$

$$f\left(x\right)g\left(x\right)-\int f'(x)g(x)\,dx=\int f(x)g'(x)\,dx$$

$$\int f(x)g'(x)\,dx=f\left(x\right)g\left(x\right)-\int f'(x)g(x)\,dx$$

So we can use this if we have some integral with a couple functions being multiplied together.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Usually we'll chose some function to be u, and another as dv.

$$\int u\,dv = uv - \int v\,du$$

Write this one down!

I like to say "ultra-violet voo-doo"

$$\int uv' = uv - \int vu'$$

Or this one (maybe a little easier to remember).

$$\int u\,dv = uv - \int v\,du$$

Let's get to work: Find $\int x \sin x \, dx$

Usually we want "u" to be something that's easy to differentiate.

Let u = x

du = dx   dv = sin x dx

v = -cos x

$$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

You can always check by finding the derivative, but this ain't no test.

Here is an easy pneumonic to help you choose which function to choose as "u". Whichever comes first in this order is the "u".

L: Logarithmic

I: Inverse Trig

A: Algebraic/(polynomial)

T: Trig

E: Exponential

Kind of like a Latte, but a little fancier.

L
I
A
T
E

Find $\int \ln x \, dx$

Well I guess it has to be...

u = ln x

du = 1/x dx   dv = dx

v = x

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx$$

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

This is actually in the back of the book if you want to get sneaky.

L
I
A
T
E

Find $\int t^2 e^t \, dt$

Since e^t is an exponential function (the bottom of LIATE), choose...

$u = t^2 \qquad v = e^t$

$du = 2t \, dt \qquad dv = e^t \, dt$

$$\int t^2 e^t \, dt = t^2 e^t - 2\int t e^t \, dt$$

Nuts! We've gotta do it again.

$u = t \qquad v = e^t$

$du = dt \qquad dv = e^t \, dt$

$$\int t^2 e^t \, dt = t^2 e^t - 2\left[t e^t - \int e^t \, dt \right]$$

$$\boxed{\int t^2 e^t \, dt = t^2 e^t - 2t e^t + 2e^t + C}$$

Find $\int e^x \sin x\, dx$

u = sin x   v = e^x

du = cos x dx   dv = e^x dx

L

I

A

T

E

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx$$

What th-? This isn't any better than what we already had!
Let's try it again...

u = cos x   v = e^x

du = -sin x dx   dv = e^x dx

$$\int e^x \sin x\, dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x\, dx \right]$$

Hey, mint!
The integrals match.

$$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

$$2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x$$

$$\boxed{\int e^x \sin x\, dx = \tfrac{1}{2}(e^x \sin x - e^x \cos x) + C}$$

Last one: Find $\int_0^1 \tan^{-1}x\,dx$

Oh yeah, it works with definite integrals too.

$u = \tan^{-1}x \quad v = x$

$du = \frac{dx}{1+x^2} \quad dv = dx$

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}(x)\Big]_0^1 - \int_0^1 x\frac{1}{1+x^2}\,dx$$

did some u-substitution here

$$\int_0^1 \tan^{-1}x\,dx = \tan^{-1}(1) - \tfrac{1}{2}\ln\left(1+x^2\right)\Big)_0^1$$

$$\int_0^1 \tan^{-1}x\,dx = \tan^{-1}(1) - \tfrac{1}{2}\ln\left(2\right) - \tfrac{1}{2}\ln\left(1\right)$$

$$\boxed{\int_0^1 \tan^{-1}x\,dx = \tfrac{\pi}{4} - \tfrac{1}{2}\ln(2)}$$

Actually, this is an integral we could have stolen from the reference page too...