6.4 Work

In the case of constant acceleration, the force is also constant and the work (W) done is defined to be the product of the force (F) and the distance (d) that the object moves.

$W = Fd$. We also need to know that

Force = (mass)(acceleration).

When Force is measured in newtons (N) and distance in meters, the unit for Work is a newton-meter, which is called a Joule (J).

When Force is measured in pounds (lbs) and distance in feet, the unit for Work is foot-pounds (ft-lb). Creative, I know.

For example:

How much work is done in lifting a 1.2 kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8 m/s^2$.

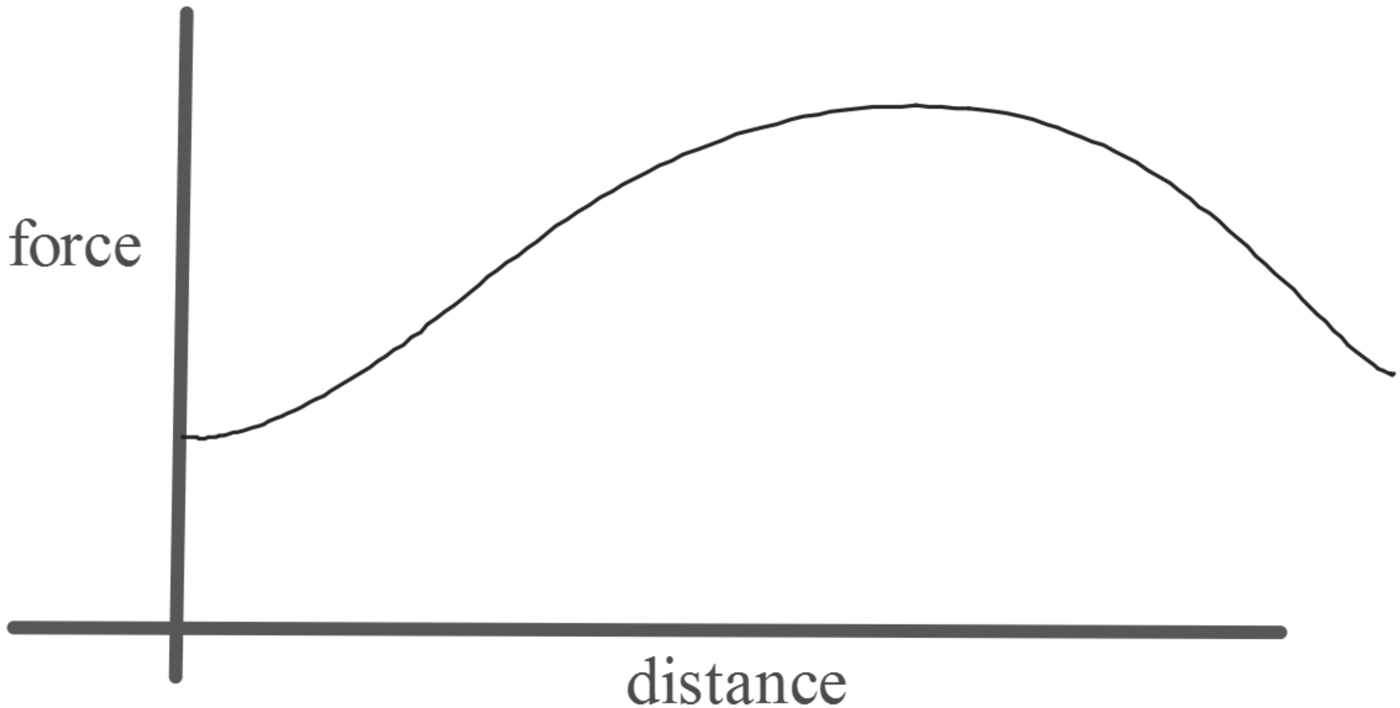
First let's find the Force (which equals mass times acceleration).

$$F = mg = (1.2)(9.8) = 11.76 \text{ N}$$

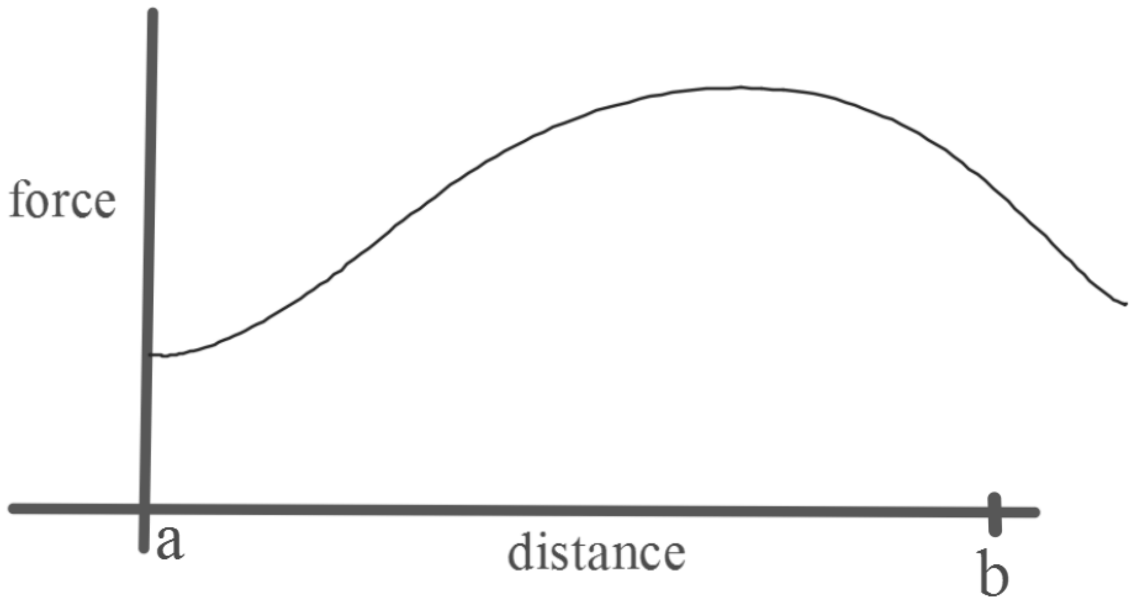
And Work?

$$W = Fd = (11.76)(0.7) = 8.2 \text{ J}$$

But what if the force is variable relative to a given distance?



Recall that Work = Force x Distance.
How could we approximate the Work done on a short interval?



How could we approximate the Work done on an entire interval, say (a,b)?

Theorem:
The work done on an interval can be found by the force function. We just need to find the area under the curve:

$$\text{Work} = \int_a^b f(x)\,dx$$

Where f(x) is the function for Force, and dx is the distance.

For Example:

When a particle is located a distance of x feet from the origin, a force of x^2+2x pounds acts on it. How much work is done in moving it from x = 1 to x = 3?

$$W = \int_1^3 \left(x^2 + 2x \right) dx$$

$$W = \left[\frac{x^3}{3} + x^2 \right]_1^3$$

$$W = \frac{50}{3}$$

The work done is $16\frac{2}{3}$ ft-lb.

Using "k" - the spring constant.

Here's a concept that pops up more than you'd think. You use Hooke's Law which equates the FORCE [f(x)] necessary to keep a spring stretched beyond it's normal length of x units.

$$f(x) = kx$$

Say you needed 40 N of force to stretch a spring past its 10cm natural length to 15cm. How much work is done in stretching the spring from 15cm to 18cm?

We use the first bit of info to get the constant "k":

$40 = 0.05k$

$k = 800$

(notice that all measurements need to be in meters - or feet if we're being archaic)

Now we integrate the product of Force and distance to get Work.

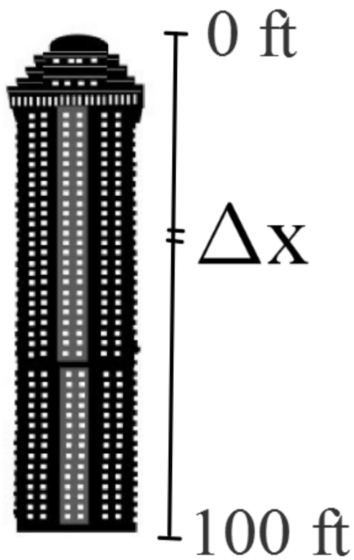
$$W = \int_{0.05}^{0.08} 800x\,dx$$

$$W = \left[400x^2\right]_{0.05}^{0.08}$$

$$W = 1.56\,\text{Joules}$$

Where did the bounds of the integral come from?

Another example: (ex 4 from your text 6.4)
A 200-lb cable is 100 feet long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?



-The cable weighs 2 lb/ft. Use 2x to represent the changing weight (the force).

-The distance is Δx (dx).

-$W = fd = 2x \cdot \Delta x$

$$W = \int_0^{100} 2x\,dx$$

$$W = \left[\frac{2x^2}{2}\right]_0^{100} = 10,000 \text{ ft-lb}$$

Work done by Pumping

When calculating how much work is done by pumping, the trick is to find the amount of force each infinitely skinny amount of material (usually water) is exerting.

Ex. How much work is required to pump a 10m high cylinder (r = 2m) containing water that is initially filled to 8m until it is half full.

You'll need to know these:

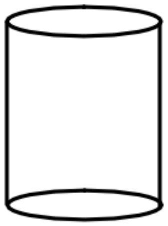
$W = Fd$

$F = ma$

$m = \text{volume} \times \text{density}$

Ex. How much work is required to pump a 10m high cylinder (r = 2m) containing water that is initially filled to 8m until it is half full.

W = Fd
F = ma
m = volume x density



The density of H_2O is 1000 kg/m³

Thinking of the volume as an infinitely skinny slice:

$v = \pi 2^2 \Delta x = 4\pi\Delta x$

$m = 4\pi\Delta x(1000) = 4000\pi\Delta x$

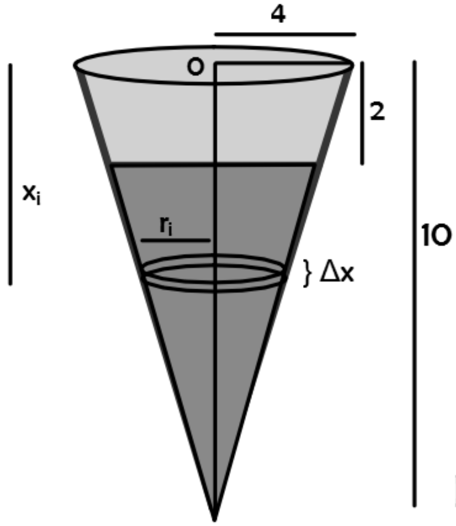
$F = 4000\pi\Delta x(9.8) = 39200\pi\Delta x$

$W = 39200\pi\Delta x(10-x) = 39200\pi\int_5^8 (10-x)dx$

$W = 411{,}600\pi$ J

Notice in these problems that the distance is the amount traveled (10-x), while the dx comes from the thickness of the slices being integrated.

Find the work required to empty this here cone. All measurements are in meters.
The density of H_2O is 1000 kg/m^3



W = Fd
F = ma
m = V x density

radius of "slice" of water: $\frac{r_i}{10-x_i} = \frac{4}{10}$ $\quad r_i = \frac{2}{5}(10-x_i)$

Volume of "slice": $V = \pi r^2 \Delta x = \frac{4\pi}{25}(10-x_i)^2 \Delta x$

mass of "slice": $m = \frac{4\pi}{25}(10-x_i)^2 \Delta x(1000) = 160\pi(10-x_i)^2 \Delta x$

Force required to deal with gravity: $F = (9.8)160\pi(10-x_i)^2 \Delta x \approx 1570\pi(10-x_i)^2 \Delta x$

Work to travel x_i:

$$W \approx F x_i \approx 1570\pi x_i (10-x_i)^2 \Delta x$$

$$W \approx \int_2^{10} 1570\pi x(10-x)^2 dx$$

$$W \approx 1570\pi\left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4}\right]_2^{10}$$

$$\boxed{W \approx 3.4 \times 10^6 J}$$