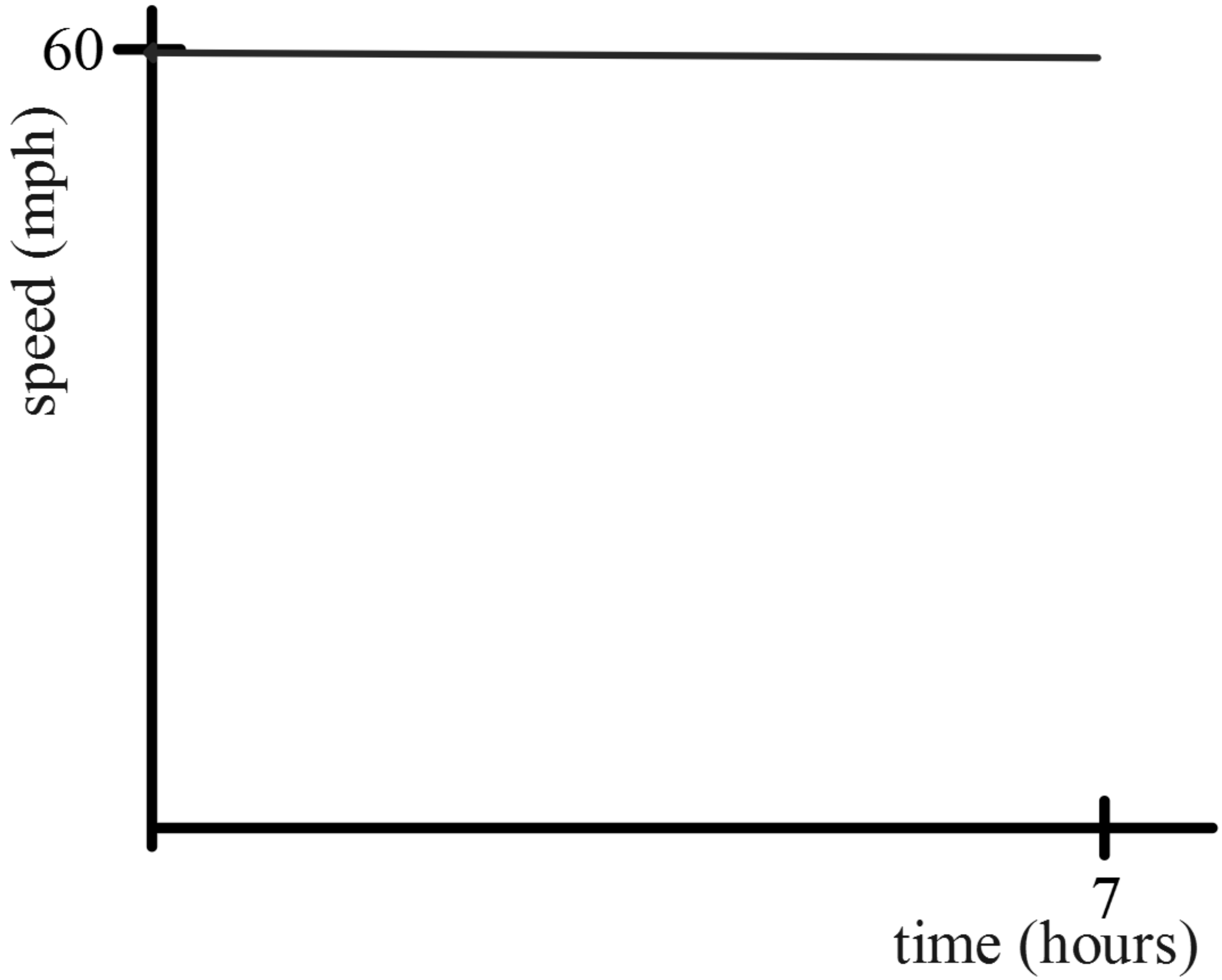
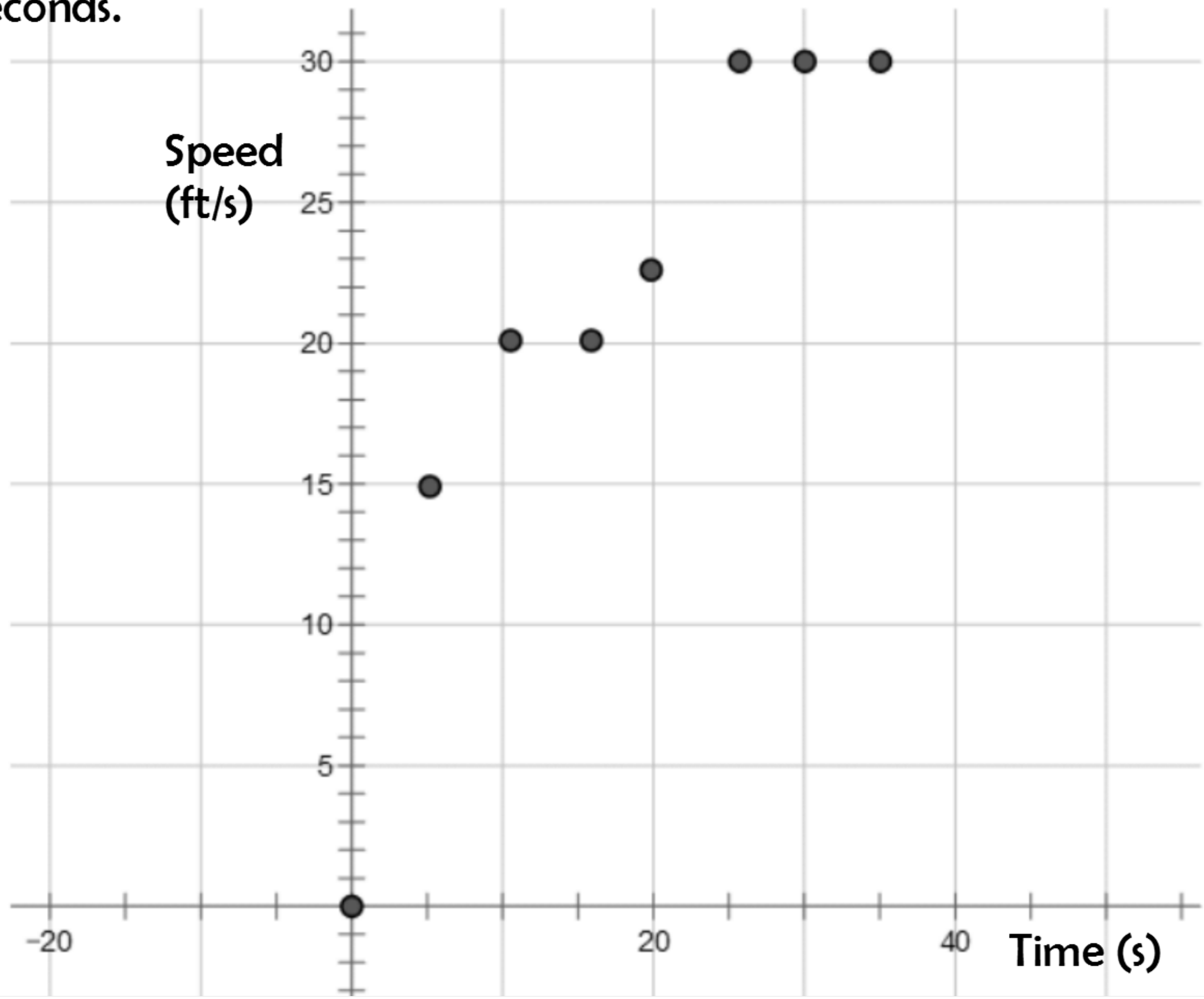
This graph describes a driving pattern. How far did the driver go?

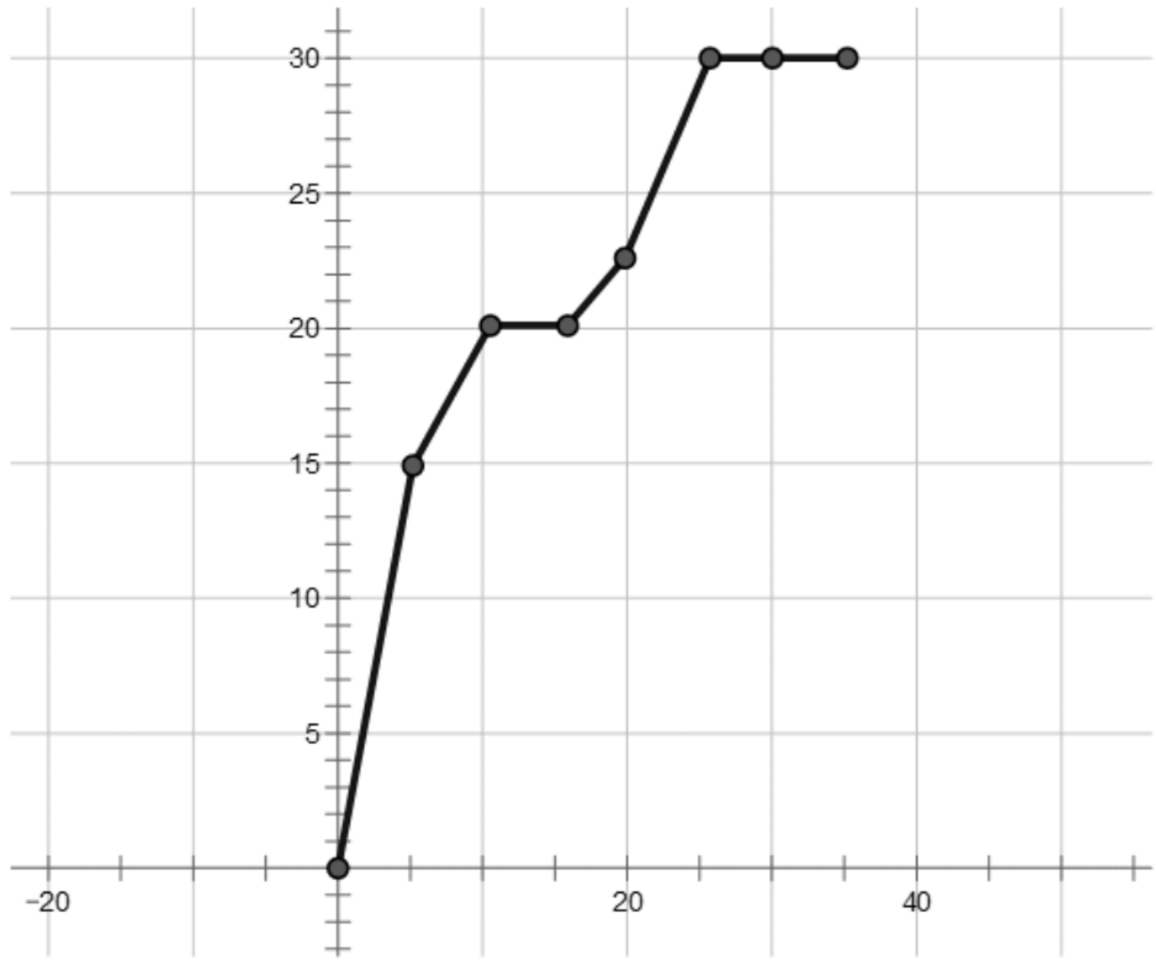


The odometer (mileage recorder) in Mr. Krasean's Dad's Oldsmobile Park Avenue is broken. In the hopes of figuring out how far he has driven, he recorded the speed his car was traveling (in feet/second) in 5 second intervals for the first 35 seconds of his trip to Maplewood Mall. Estimate how far he traveled in those 35 seconds.



5.1 - Areas and Distances

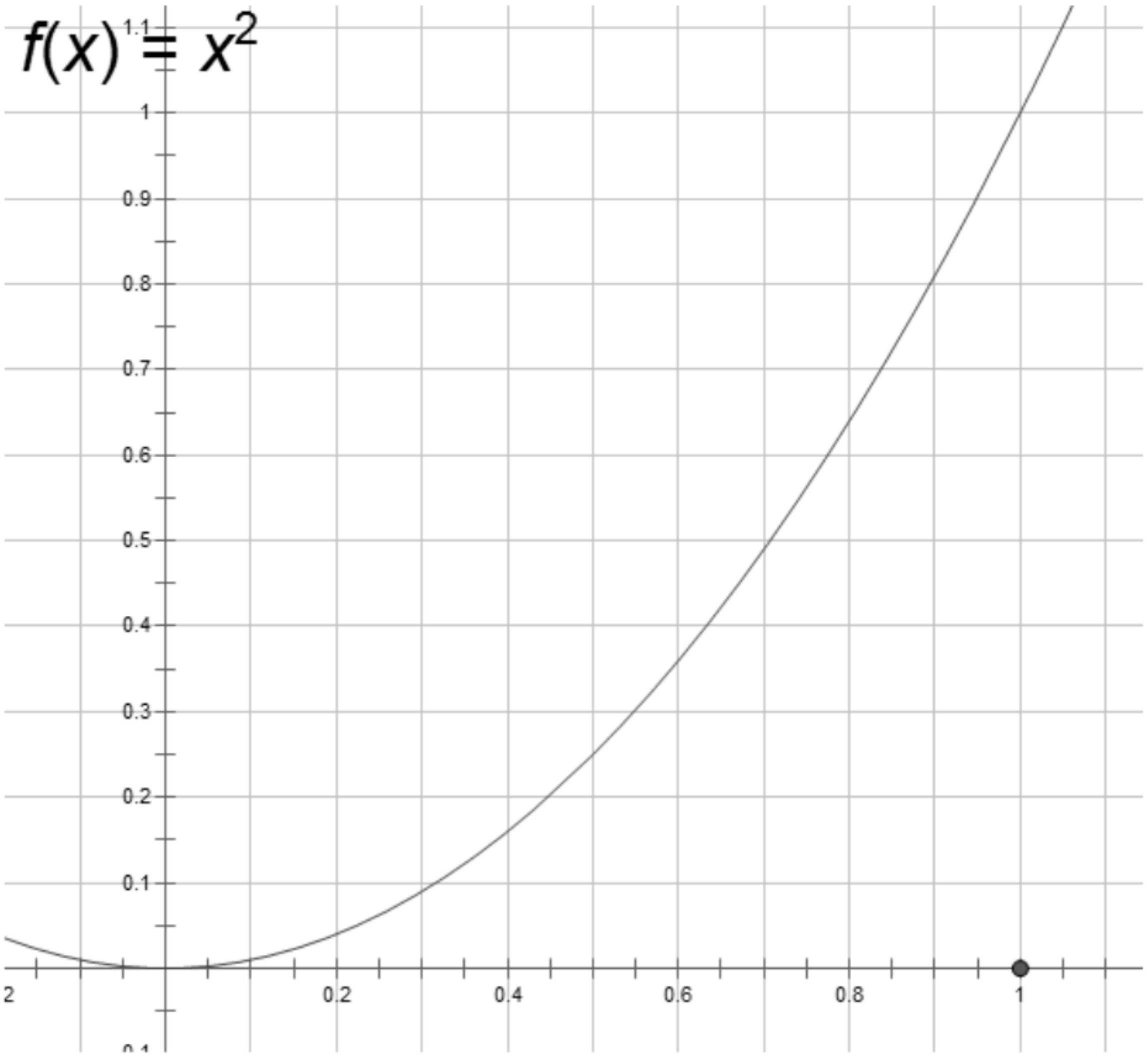
Find the area bound by the function f, the x axis, and the line x=35.



$$5(0+15)/2 + 5(15+20)/2 + 5(20) + 5(20+22.5)/2 + 5(22.5+30)/2 + 5(30) + 5(30) = 762.5$$

What do we notice about the area bound under the curve and the distance traveled?

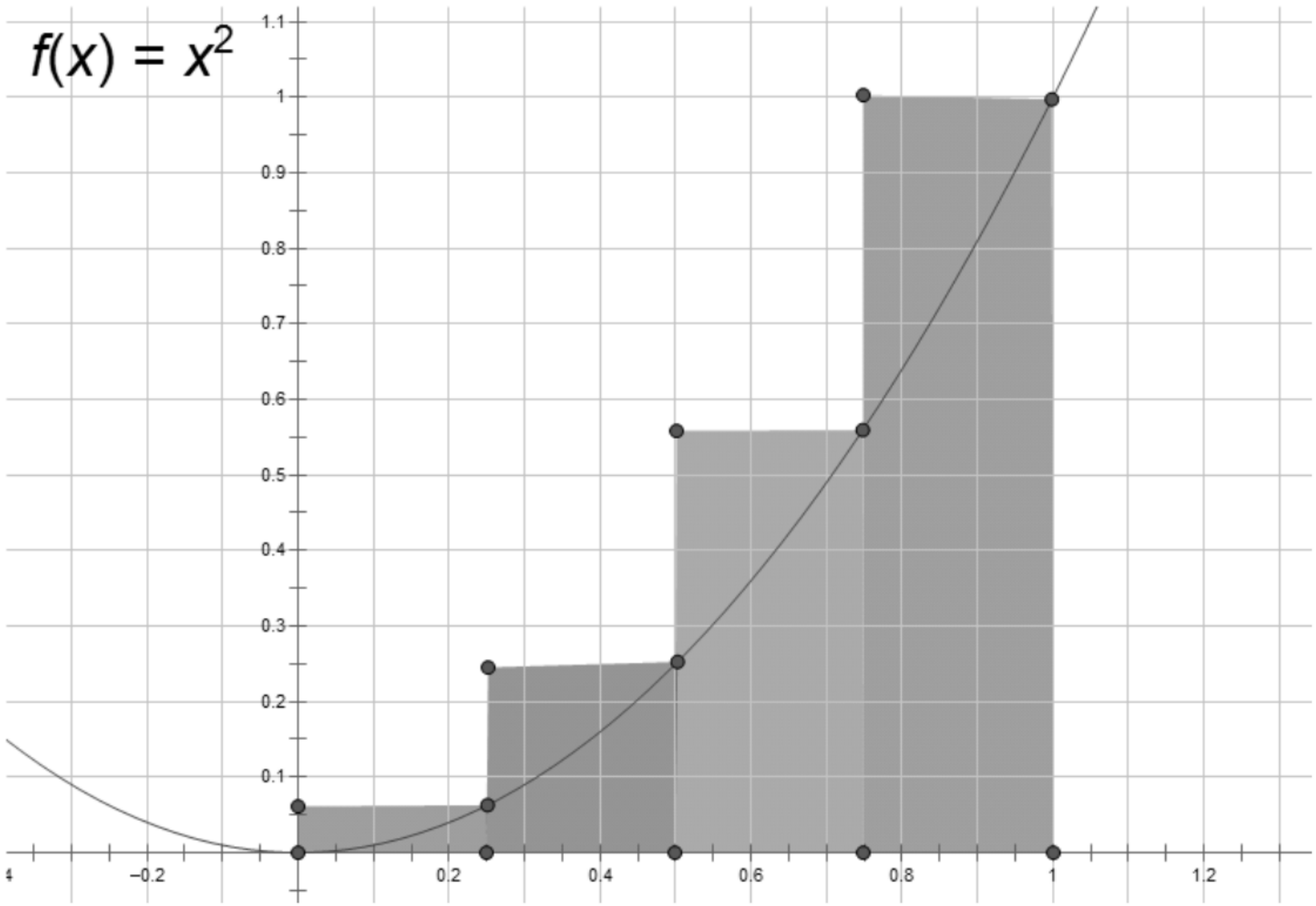
That one was kind of easy because we had straight lines. What if I wanted to estimate the area under this curve on the interval [0,1]?



So here is one estimate. It is clearly an overestimate, right?
We can see that

$R_4 = 0.25(0.25)^2 + 0.25(0.5)^2 + 0.25(0.75)^2 + 0.25(1)^2$

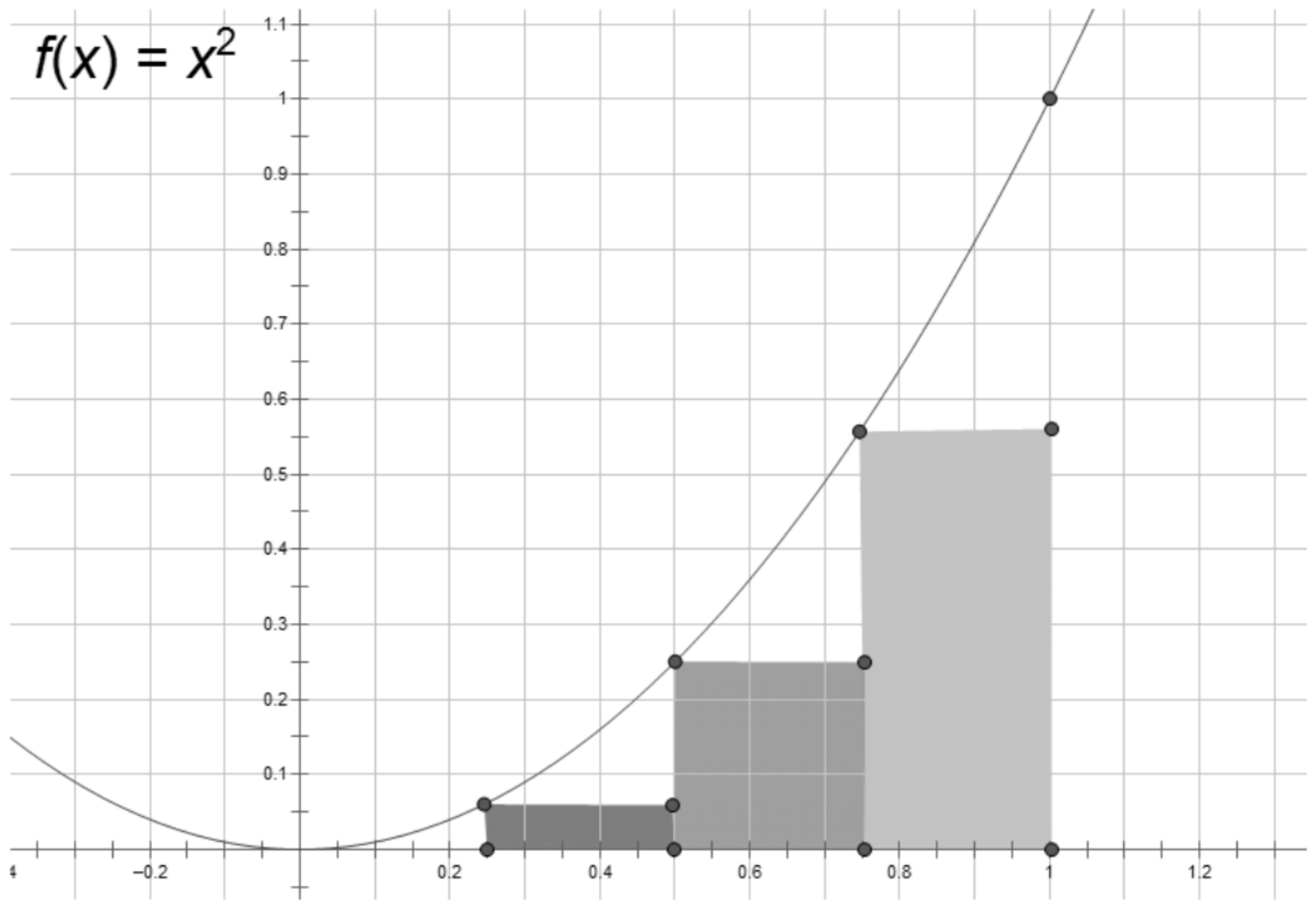
$R_4 = 0.468575$



Use R_n when using the function to find the <u>right</u> side of the rectangle.

n = number of rectangles

Here is another estimate. This is an underestimate.
In this case we have:
$L_4=0.25(0)^2+0.25(.25)^2+0.25(0.5)^2+0.25(0.75)^2$
$L_4=0.21875$



Use L_n when using the function to find the left side of the rectangle.

n = number of rectangles

We've already done left and right endpoint approximations.
Here's the Midpoint Rule:

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + ... + f(\bar{x}_n)]$$

$$\text{where } \Delta x = \frac{b-a}{n}$$

$$\text{and } \bar{x}_i = \tfrac{1}{2}(x_{i-1} - x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Here's a midpoint problem to try:
Use the midpoint rule with n=5 to approximate $\int_1^2 \frac{1}{x}\,dx$

Since n=5, $\Delta x = (2-1)/5 = 0.2$

This means each "box" has a width of 0.2, and they will start at x = 1, 1.2, 1.4, 1.6, and 1.8.

The midpoints will be the averages of each interval, so we'll need f(1.1), f(1.3), f(1.5), f(1.7), and f(1.9).

$$M_5 = \frac{1}{5}\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]$$

$$M_5 = \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right]$$

$$M_5 \approx 0.691908$$

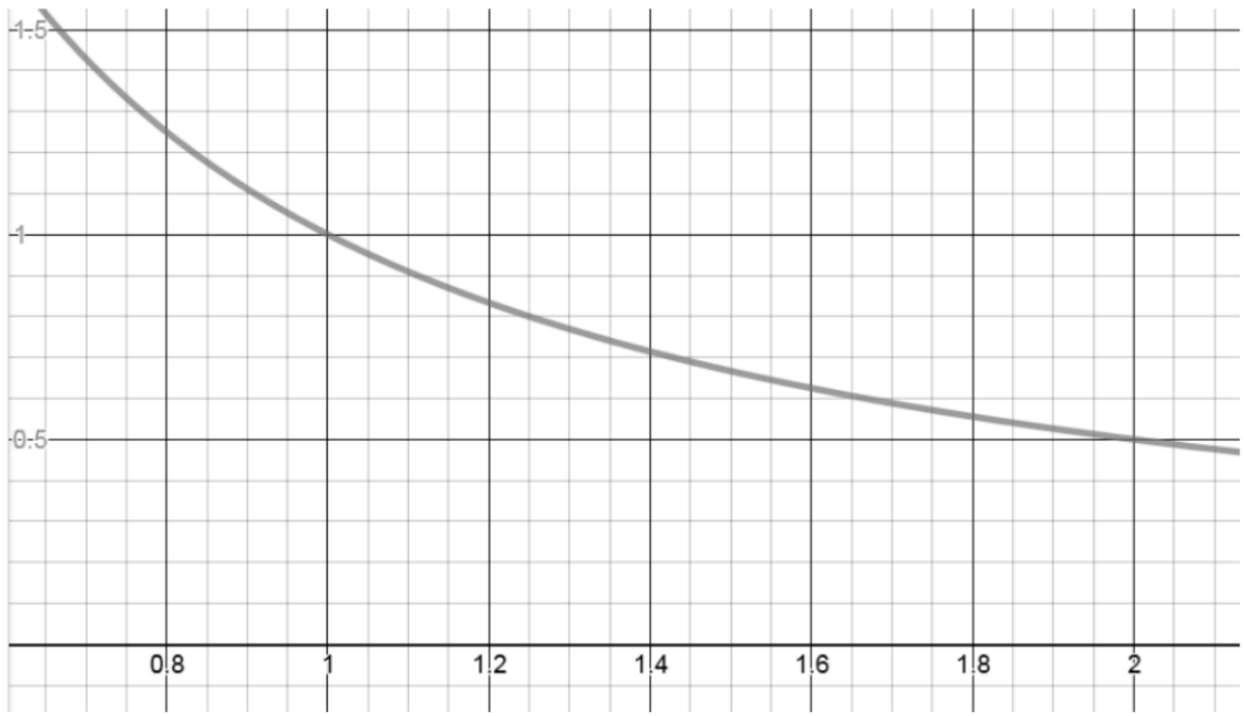
The Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n$$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right]$$

where $\Delta x = \frac{b-a}{n}$

and $x_i = a + i\Delta x$



We'll try that same integral with the trapezoidal rule.
Use the trapezoidal rule with n=5 to approximate $\int_1^2 \frac{1}{x}\,dx$

Since n=5, Δx = (2-1)/5 = 0.2

Same as with the midpoint rule, each "box" has a width of 0.2, and they will start at x = 1, 1.2, 1.4, 1.6, and 1.8.

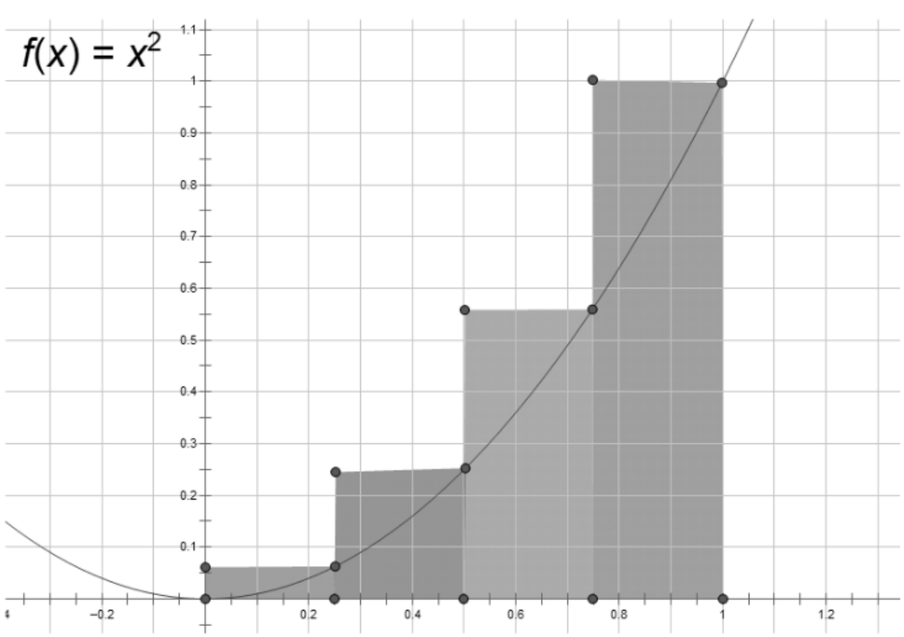
$$T_5 = \frac{1}{5(2)}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$$T_5 = \frac{1}{10}\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right]$$

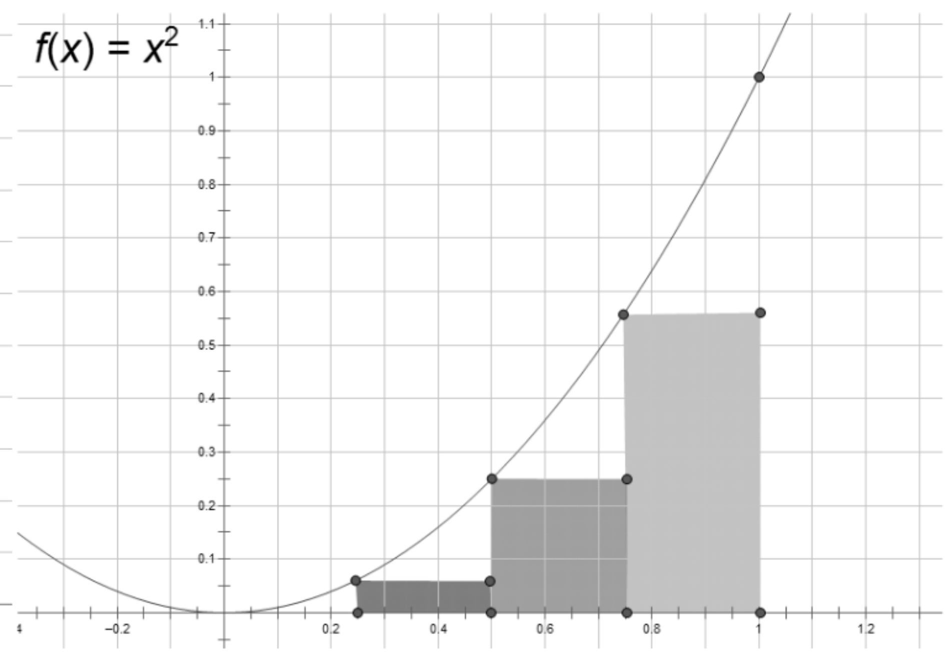
$$T_5 \approx 0.695635$$

Here are two estimates for the area under the curve on [0,1]. We are not extremely close on either, but we can see that the actual area lies somewhere between the overestimate on the left and the underestimate on the right.

$R_4=0.4875$



$L_4=0.21875$



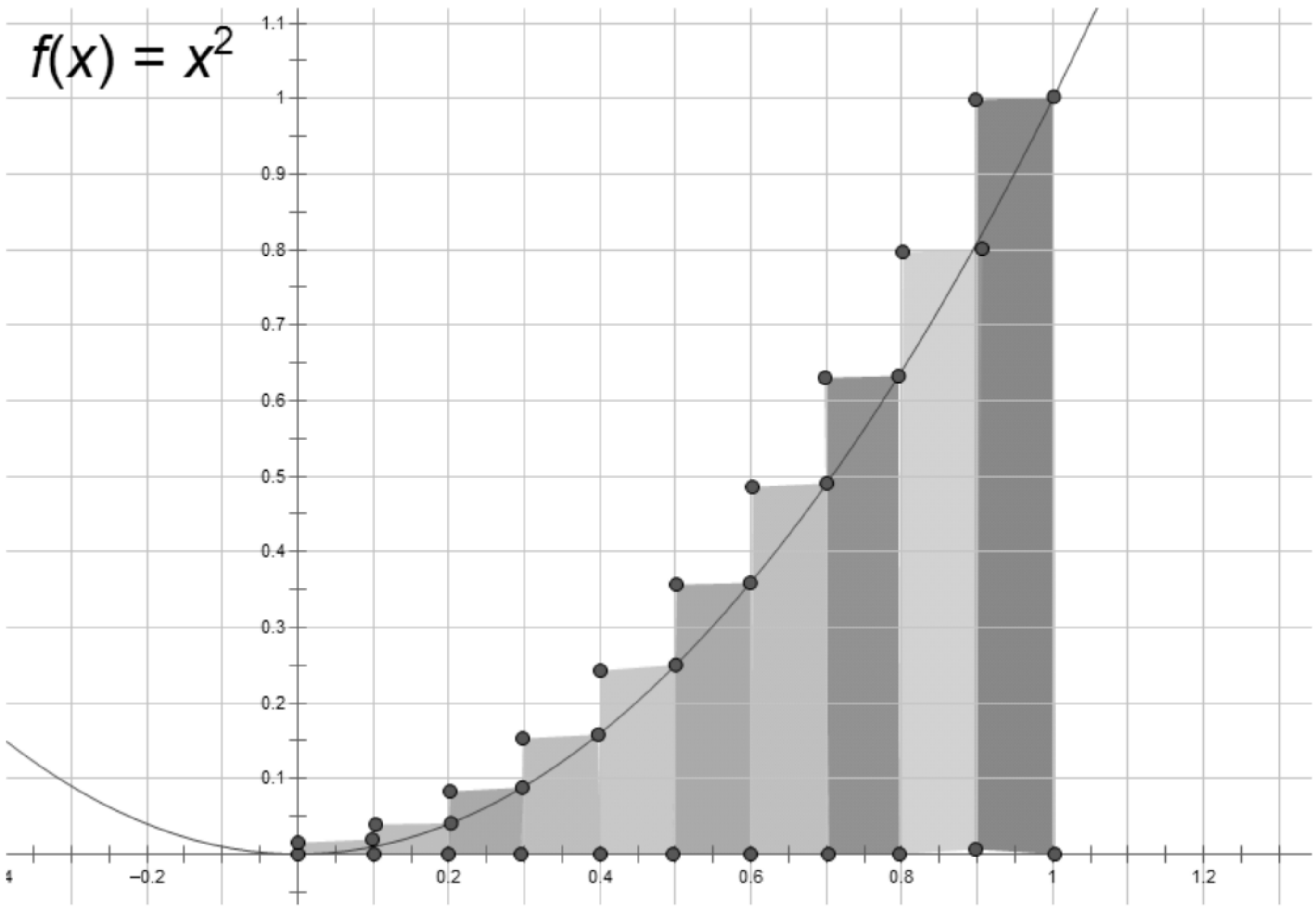
So, when n=4: $0.21875<A<0.4875$

Let's get more accurate. What if, instead of 4 rectangles, we were able to use 10? This would result:

$R_{10}=0.1(0.1)^2+0.1(0.2)^2+0.1(0.3)^2+\ldots+0.1(0.9)^2+0.1(1)^2$

$R_{10}=0.385$

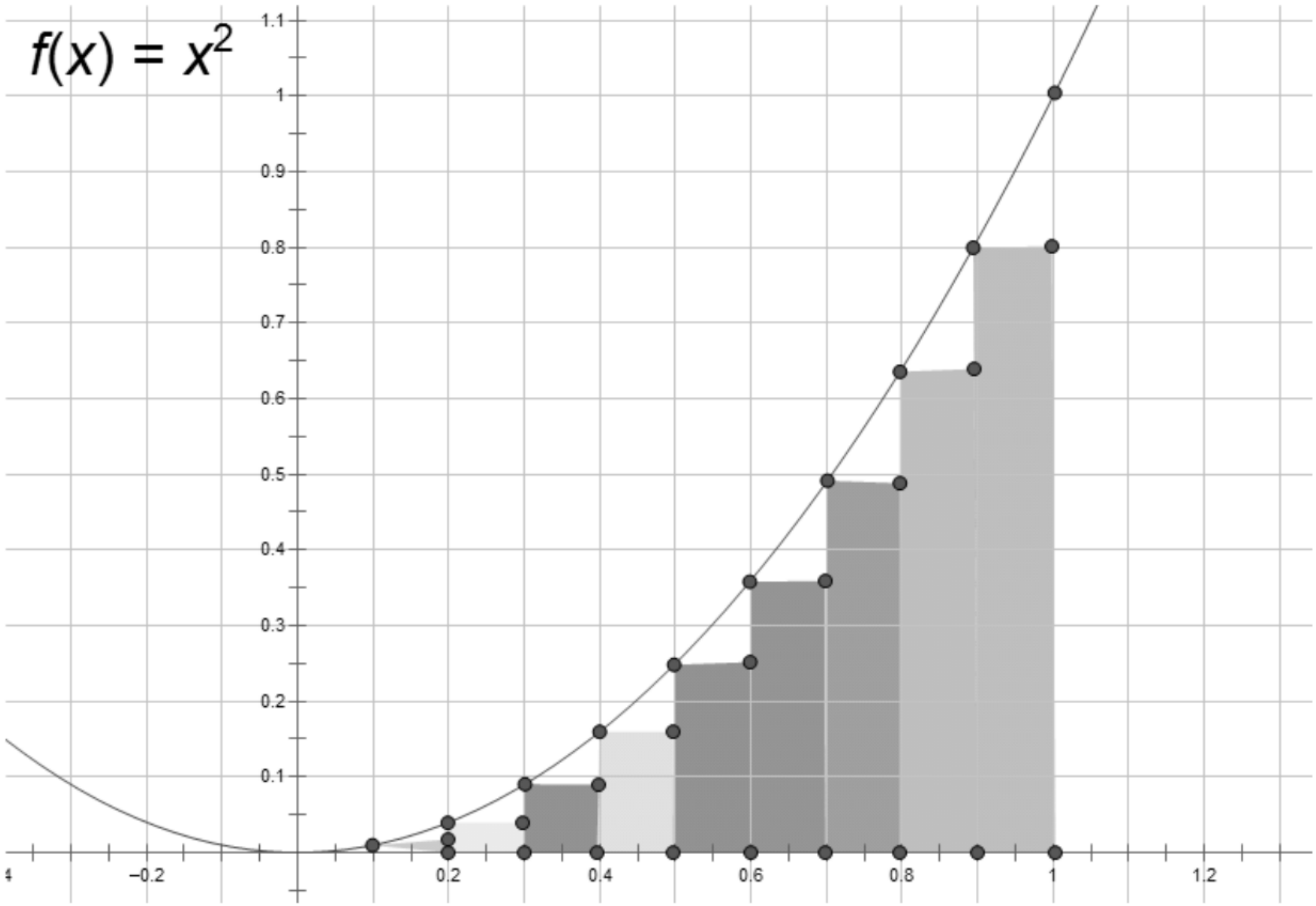
Again, this is an overestimate.



This underestimate would result in:

$L_{10}=0.1(0)^2+0.1(0.1)^2+0.1(0.2)^2+0.1(0.3)^2+\ldots+0.1(0.9)^2$

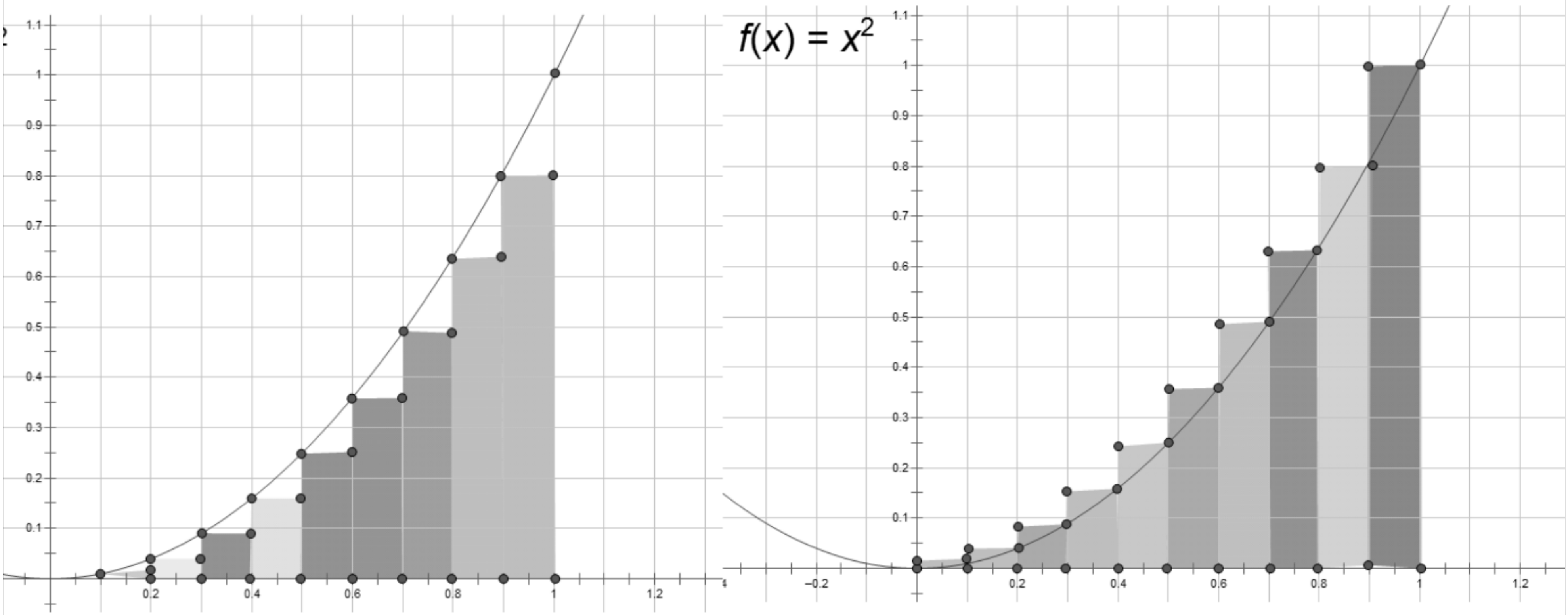
$L_{10}=0.285$



So these two area approximations, when n=10, allow us to estimate the true area under the curve to:

$0.285<A<0.385$

Compare this possible range with what we had when n=4: $0.21875<A<0.4875$



We are getting closer and closer to the actual value:

When n = 4: $0.21875<A<0.4875$

When n = 10: $0.285<A<0.385$

When n = 30: $0.3169<A<0.3502$

When n = 50: $0.3234<A<0.3434$

When n = 100?

When n = 1,000,000?

A when n goes to infinity?

Any Guesses?

Riemann Sums

n : number of rectangles

R_n : n rectangles where the height of the rectangle is determined by the value (height) of the function at the <u>right</u> side of the rectangle

L_n : n rectangles where the height of the rectangle is determined by the value (height) of the function at the <u>left</u> side of the rectangle

M_n : n rectangles where the height of the rectangle is determined by the value (height) of the function in the <u>middle</u> of the rectangle

T_n : n trapezoids where the height of the each side is determined by the value (height) of the function at both sides of the trapezoid.

Riemann Sums

n : number of rectangles

R_n : n rectangles where the height of the rectangle is determined by the value (height) of the function at the right side of the rectangle

L_n : n rectangles where the height of the rectangle is determined by the value (height) of the function at the left side of the rectangle

M_n : n rectangles where the height of the rectangle is determined by the value (height) of the function in the middle of the rectangle

T_n : n trapezoids where the height of the each side is determined by the value (height) of the function at both sides of the trapezoid.

The Midpoint Rule:

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)]$$

$$\text{where } \Delta x = \frac{b-a}{n}$$

$$\text{and } \bar{x}_i = \frac{1}{2}(x_{i-1} - x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

The Trapezoidal Rule:

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$$T_n = \frac{\Delta x}{2}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + \ldots + 2f\left(x_{n-1}\right) + f\left(x_n\right)\right]$$

$$\text{where } \Delta x = \frac{b-a}{n}$$

$$\text{and } x_i = a + i\Delta x$$

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$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)]$$

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$$\text{where } \Delta x = \frac{b-a}{n}$$

$$\text{and } x_i = a + i\Delta x$$