Opener

Find the equation of the line tangent to the function $y = 3xe^{-2x}$

at the point $\left(1, \frac{3}{e^2}\right)$.

1. The equation is y - $\frac{3}{e^2}$ = m(x - 1)

   we need to find m.

2. m is y' when x = 1

3. $y' = -6xe^{-2x} + 3e^{-2x}$

4. $y'\left(1\right) = \frac{-3}{e^2}$

5. So the answer is $\boxed{y - \frac{3}{e^2} = -\frac{3}{e^2}(x-1)}$

Calculus opener

If $x^2 + xy = 10$, find $\frac{dy}{dx}$ when $x = 2$.

$$-\frac{7}{2}$$

Use Liebniz's Notation (dy/dx) to find the derivative of the function:

$$y = (3x^2 + \sqrt{x})^5$$

$$\frac{dy}{dx} = 5\left(3x^2 + \sqrt{x}\right)^4\left(6x + \frac{1}{2\sqrt{x}}\right)$$

Really what we're doing when we take a derivative is using the Chain Rule every single time.

taking a derivative | chain rule | simplifying $\left(\frac{dx}{dx} = 1\right)$

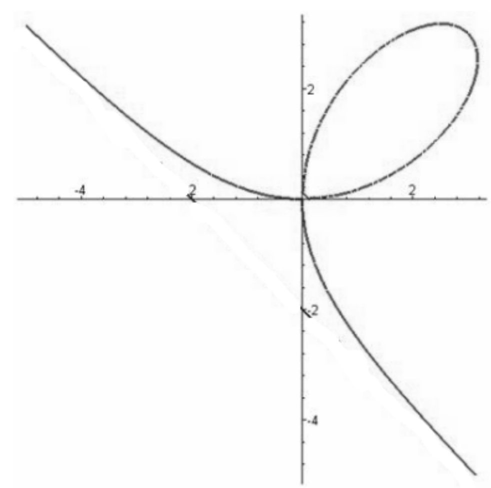
$$\frac{d}{dx}x^2 = 2x^1\left(\frac{d}{dx}x\right) = 2x\left(\frac{dx}{dx}\right) = 2x$$

$$\frac{d}{dx}y^5 = 5y^4\left(\frac{d}{dx}y\right) = 5y^4\left(\frac{dy}{dx}\right) = 5y^4\frac{dy}{dx}$$

This concept is a big deal when we're dealing with functions where we can't (easily) solve for y.

Take this thing called the "folium of Descartes"

$$x^3 + y^3 = 6xy$$



I don't think I could be less interested in solving this for y. But we can still differentiate it...

First off, let's say that we can usually write a function "explicitly" (exactly specifically?) in terms of another variable.

$$2x + 3y = 9$$

$$y = -\frac{2}{3}x + 3$$

If we wanted to differentiate it, that'd be all kinds of easy

$$\frac{dy}{dx} = -\frac{2}{3}$$

This means the derivative of y in terms of x.

Rather than solving nastier equations for y (like the folium of Descartes, or an ellipse, or most other conics), we can treat them like that old turn-of-the-century phrase "let's not and say we did". It's "implied" that we *could* solve for y. But we won't.

Although we can still differentiate it! You just need:
1. Liebniz' notation
2. The Chain Rule
3. Some seriously mad algebra skillz.

It's a process called Implicit Differentiation. So pull up your big kid pants and let's tame this beast.

Derive: $x^3 + y^3 = 6xy$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}6xy$$

Take the derivative of both sides

$$3x^2\frac{dx}{dx} + 3y^2\frac{dy}{dx} = 6\left(x\frac{dy}{dx} + \frac{dx}{dx}y\right)$$

These are dumb to write, but I'm just emphasizing that we're using the Chain Rule

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(x\frac{dy}{dx} + y\right)$$

Cancel the easy 1's (this is usually what you'd write first)

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

Distribute (in this case at least)

$$\left(3y^2 - 6x\right)\frac{dy}{dx} = 6y - 3x^2$$

Now you're just solving for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

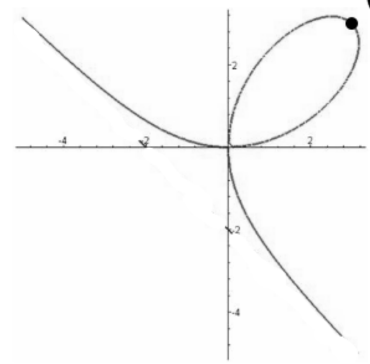
OK, but factor those 3's to make it pretty

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Yahtzee - this is derivative of y in terms of x.

Let's think about what we have. Say you wanted the equation of the tangent line of this 'folium of Descartes' at (3, 3).

Visual aide



Original deal:

$$x^3 + y^3 = 6xy$$

Derivative:

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

"Just" put in x=3 and y=3 into $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2(3) - 9}{9 - 2(3)}$$

$$\frac{dy}{dx} = \frac{-3}{3} = -1$$

← This looks like a reasonable slope at (3, 3)

y - 3 = -1(x - 3)
y = -x + 6
x + y = 6

← Any of these are exceptional, acceptable answers

Try using implicit differentiation on this:

$$\sin(x+y) = y^2 \cos x$$

$$\cos(x+y)\frac{d}{dx}(x+y) = y^2\frac{d}{dx}\cos x + \cos x\frac{d}{dx}y^2$$

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) = y^2(-\sin x) + \cos x(2y)\frac{dy}{dx}$$

$$\cos(x+y) + \frac{dy}{dx}\cos(x+y) = -y^2(\sin x) + \frac{dy}{dx}\cos x(2y)$$

$$\frac{dy}{dx}\cos(x+y) - \frac{dy}{dx}\cos x(2y) = -y^2(\sin x) - \cos(x+y)$$

$$\frac{dy}{dx}(\cos(x+y) - 2y\cos x) = -y^2(\sin x) - \cos(x+y)$$

$$\frac{dy}{dx} = \frac{-y^2(\sin x) - \cos(x+y)}{\cos(x+y) - 2y\cos x}$$

$$\frac{dy}{dx} = \frac{y^2(\sin x) + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Find y" - you'll want to use implicit differentiation on this too.

$$9x^2 + y^2 = 9$$

$$18x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-18x}{2y}$$

$$\frac{dy}{dx} = -9\frac{x}{y}$$

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(-9\frac{x}{y}\right)$$

$$\frac{d^2y}{dx^2} = -9\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

If you're lazy you can write y" instead of $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = -9\left(\frac{y - x\left(-9\frac{x}{y}\right)}{y^2}\right)$$

Since $\frac{dy}{dx} = -9\frac{x}{y}$

$$\frac{d^2y}{dx^2} = -9\left(\frac{y^2 + 9x^2}{y^3}\right)$$

Since $9x^2 + y^2 = 9$

$$\frac{d^2y}{dx^2} = -9\left(\frac{9}{y^3}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-81}{y^3}$$

We use implicit differentiation to find the derivatives of inverse trigonometric functions.

Take our friend arcsin (if $y = \sin\theta$, then $\theta = \sin^{-1} y$)

Say we had $\boxed{y = \sin^{-1}x}$ as our function

Then $x = \sin y$, and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Differentiating with respect to x:

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Since

$$\cos y = \sqrt{1-\sin^2 y} \quad \text{(Pythagorean Identity)}$$

$$= \sqrt{1-x^2} \quad (x = \sin y)$$

Then $\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$

So $\boxed{\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$

A list of your inverse trig derivatives.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$