11.9: Representations of Functions as Power Series.

Remember that a geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$

is convergent if $|x| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

This means we could write $\frac{a}{1-r}$ as a power series.

Why would we ever want to do that? Well, sometimes $\frac{a}{1-r}$, or at least something close to this form, is really tough to integrate or differentiate. As a power series, however, well those are easy to integrate and differentiate.

Express as the sum of a power series and find the interval of convergence.

$$\frac{1}{1+x^2}$$

$$\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)}$$

$$\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)} = \sum_{n=0}^{\infty}\left(-x^2\right)^n$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

Write as a geometric series, where $r = -x^2$

This converges when $|-x^2| < 1$, or just $x^2 < 1$, or $-1 < x < 1$.
So the interval of convergence is $(-1, 1)$

Express as the sum of a power series and find the interval of convergence.

$$\frac{1}{2+x}$$

Write as a geometric series, where $r = -\frac{x}{2}$

$$\frac{1}{2+x} = \frac{1}{2(1-(-\frac{x}{2}))}$$

$$\frac{1}{2+x} = \frac{1}{2(1-(-\frac{x}{2}))} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This converges when |-x/2| < 1, or just |x| < 2, or -2 < x < 2.
So the interval of convergence is (-2, 2).

Express as the sum of a power series and find the interval of convergence.

$$\frac{x^3}{x+2}$$

Write as a geometric series, where $r = -\frac{x}{2}$

$$\frac{x^3}{x+2} = x^3 \frac{1}{2(1-(-\frac{x}{2}))}$$

$$x^3 \frac{1}{x+2} = x^3 \frac{1}{2(1-(-\frac{x}{2}))} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}$$

$$= 4\sum_{n=3}^{\infty} \left(-\frac{x}{2}\right)^n$$

$$= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \ldots$$

This converges when $|-x/2| < 1$, or just $|x| < 2$, or $-2 < x < 2$.
So the interval of convergence is $(-2, 2)$.

1. $f(x) = \frac{1}{1+x^4}$

2. $f(x) = \frac{1}{1+4x^2}$

Find a power series representation of the following.

a. Write the first 4 terms
b. Write the general term
c. Represent the power series in sigma notation
d. Determine the interval of convergence
e. Find the radius of convergence

Theorem:

If the power series

$$\sum_{n+1}^{\infty} c_n (x-a)^n$$

has radius of convergence $R > 0$

then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \Sigma_{n+1}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval (a-R, a+R) and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$

$$= \Sigma_{n=0}^{\infty} c_{n+1}(n+1)(x-a)^n$$

$$= \Sigma_{n=1}^{\infty} c_n (n)(x-a)^{n-1}$$

(ii) $\int f(x)\,dx = C + c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3$

$$= C + \Sigma_{n=0}^{\infty} \square$$

The radius of convergence of the power series in Equations (i) and (ii) are both R

Example:
Find a power series representation for

$$\ln(1-x)$$

and its radius of convergence.

$$\frac{d}{dx}\ln\left(1-x\right) = (-1)\frac{1}{1-x} \qquad r = x$$

$$-1\Sigma_{n=0}^{\infty}x^{n} = -1 - x - x^{2} - \ldots$$