10.6 - Absolute Convergence and the Ratio Test

The series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

The series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if it is convergent but

$\sum_{n=1}^{\infty} |a_n|$ is **not** convergent.

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, it is also convergent.

Test for absolute convergence if you have a series where the terms have negatives, but don't strictly alternate signs.

The series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

We can tell this is convergent by:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

p-series; p=2 > 1 ,
so convergent

Since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right|$ is convergent, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent,

and therefore **convergent**.

The series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if it is convergent but

$\sum_{n=1}^{\infty} |a_n|$ is **not** convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ This series (the alternating harmonic) is conditionally convergent.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic, so divergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent (by the Alternating Series Test), so the series is only conditionally convergent.

Example - is the following series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1} + \frac{\cos 2}{4} + \frac{\cos 3}{9} + \frac{\cos 4}{16} + \frac{\cos 5}{25} \cdots$$

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This series changes signs throughout, but not in an alternating manner (so we can't use an Alternating Series Test).

Check to see if it is absolutely convergent:

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ convergent (p-series; p=2 > 1)

So, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ is convergent by the Comparison Test

And thus $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and therefore convergent.

Ratio Test

This test is nice for determining if a given series is absolutely convergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ is absolutely convergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum a_n$ is divergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then the ratio test is inconclusive.

Example: Determine if the following series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} \frac{3^n}{(-1)^n n^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{-1}{3} \left(\frac{n+1}{n} \right)^3 \right| = \lim_{n\to\infty} \left| \frac{-1}{3} \left(\frac{1+\frac{1}{n}}{1} \right)^3 \right| = \frac{1}{3} < 1$$

The series is absolutely convergent
(and therefore convergent) by the ratio test

Example: Determine if the following series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{1}{n+1} \frac{n+1}{1} \left(\frac{n+1}{n} \right)^n \right| = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = e > 1$$

Since e > 1, the series diverges by the Ratio Test