10.5 - Alternating Series

While the comparison tests from before only dealt with series that have positive terms, some series have terms that are not always positive.

Alternating Series - a series where the terms alternate between positive and negative values.

Find the first FOUR terms of each series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \ldots$$

Alteration of signs can come from...

$(-1)^{n-1}$	-1 for even n; 1 for odd n
$(-1)^{n}$	-1 for odd n
$(-1)^{n+1}$, etc.	-1 for even n
$\cos(n\pi)$	-1 for odd n

So look for these in your series.

The Alternating Series Test

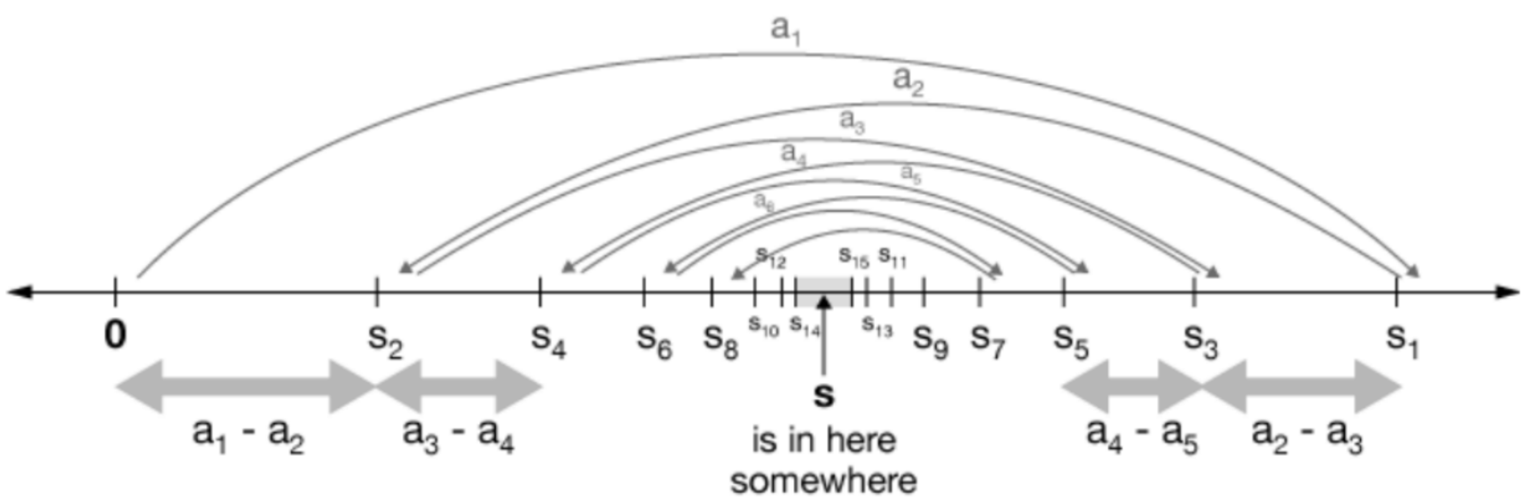
If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots$$ where $b_n > 0$

Satisfies

- $b_{n+1} \leq b_n$ for all n
- $\lim_{n\to\infty} b_n = 0$

then the series is convergent.



a1
a2
a3
a4
a5
a6
s12
s15
s11
0
s2
s4
s6
s8
s10
s14
s13
s9
s7
s5
s3
s1
a1 - a2
a3 - a4
s
is in here
somewhere
a4 - a5
a2 - a3

Ex. The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

$$b_n = \frac{1}{n} > 0$$

$$b_{n+1} < b_n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

So the series is **convergent** by the Alternating Series Test.

So while the harmonic series diverges (as shown by the integral test), the alternating harmonic series converges.

Ex. The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ alternates but

$$b_n = \frac{3n}{4n-1} > 0$$

$$b_{n+1} < b_n \qquad \text{ex.} \left[b_2 = \frac{6}{7} < b_1 = \frac{3}{3} \right]$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$$ Fail!

Instead, look at the limit of the nth term:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$ which does not exist (since $\lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$ oscillates from $\frac{-3}{4}$ to $\frac{3}{4}$)

so we can conclude that the series **diverges** by the nth term test.

Try this one - is the series converging or diverging?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$$

Is b_n positive?

Is b_n decreasing? Hint: find the derivative!

Does $\lim_{n\to\infty} b_n = 0$?

If we're able to use the Alternating Series Test, we can estimate the total sum (using a partial sum) and find the amount of error (which is the same as the n+1 term).

If $s = \Sigma(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$0 \leq b_{n+1} \leq b_n \qquad \text{and} \qquad \lim_{n \to \infty} b_n = 0$$

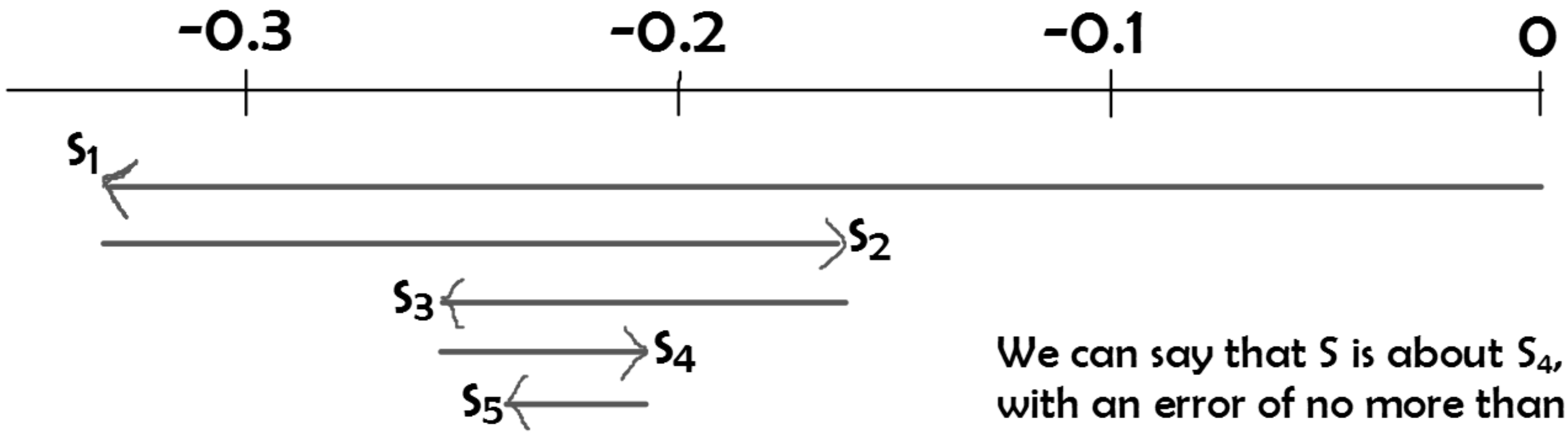
then $$|R_n| = |s - s_n| \leq b_{n+1}$$

- $|R_n|$: Remainder (the amount or error)
- s: The 'real' sum
- s_n: The partial sum
- b_{n+1}: The first 'neglected' term

So say we had the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2+2}$

$$s = -\frac{1}{3} + \frac{1}{6} - \frac{1}{11} + \frac{1}{18} - \frac{1}{27} \cdots$$

$$s_1 = -0.\overline{333}$$
$$s_2 = -0.\overline{333} + 0.1\overline{66} = -0.1\overline{66}$$
$$s_3 = -0.\overline{333} + 0.1\overline{66} - 0.\overline{09} = -0.257$$
$$s_4 = -0.\overline{333} + 0.1\overline{66} - 0.\overline{09} + 0.0\overline{55} = -0.202$$
$$s_5 = -0.\overline{333} + 0.1\overline{66} - 0.\overline{09} + 0.0\overline{55} - 0.\overline{037} = -0.239$$



We can say that S is about S_4, with an error of no more than S_5.

Say, from our previous example, we wanted to find how many terms we'd need to add to get an approximation for the sum of the series where the error is no more than 0.01.

Since the 'remainder', or error, is the first unused term, just find which term is first below 0.01.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2+2}$$

$$a_1 = -\frac{1}{3} \qquad a_6 = \frac{1}{38}$$

$$a_2 = \frac{1}{6} \qquad a_7 = -\frac{1}{51}$$

$$a_3 = -\frac{1}{11} \qquad a_8 = \frac{1}{66}$$

$$a_4 = \frac{1}{18} \qquad a_9 = -\frac{1}{83}$$

$$a_5 = -\frac{1}{27} \qquad a_{10} = \frac{1}{102}$$

Since $a_{10} < 0.01$,
we need to add the first 9 terms to get an approximation within the required error.

Finding partial sums in the TI-84

https://www.youtube.com/watch?v=JCN4VK4Tojg