10.4 - Comparison Tests

Comparison tests are used to compare a series to one that is known to be convergent or divergent.

Ex. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ Convergent or Divergent?

Since $\frac{1}{2^n+1} < \frac{1}{2^n}$, and we already know

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ to be convergent (it's geometric with $r = 1/2 < 1$)

it follows that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ must also be convergent.

The Comparison Test

Suppose Σa_n and Σb_n are series with positive terms,

- If Σb_n is convergent and $a_n \leq b_n$ for all n,

  Σa_n is convergent.

- If Σb_n is divergent and $a_n \geq b_n$ for all n,

  Σa_n is divergent.

Ex. Does the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converge?

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

Choose a similar, simpler function to compare. The given one is almost a p-series with p=2, and we know that converges. Right?

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

convergent (p=series p = 2 > 1)

So by the Comparison Test, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges.

Ex. Does the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

$$\frac{\ln n}{n} > \frac{1}{n}$$

Here a good idea is to choose the harmonic series - one we know to diverge, and the given series is larger.

$\sum_{n=1}^{\infty} \frac{1}{n}$ Divergent - harmonic

So by the Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

To solve these, you'll need:

- An inequality of your comparison
- How you know your comparison converges or diverges
- The statement of using the Comparison Test to determine the answer

Ex. Does the series $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$ converge?

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

Our given series is almost geometric, so compare it to an easier version that actually is geometric.

$\sum_{n=1}^{\infty}\frac{1}{2^n}$ Is geometric, with $r = \frac{1}{2} < 1$ so it converges.

$$\frac{a_1}{1-r}$$

So by the Comparison Test, $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$ converges.

The Limit Comparison Test

Suppose Σa_n and Σb_n are series with positive terms,

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where c is finite and c > 0

then both series with converge or diverge.

Here's why the Limit Comparison Test works:

Ex. Does the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge?

Let $a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n}$ Again, choose something similar and simpler.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n - 1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges - it is geometric with $r = \frac{1}{2} < 1$

then by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

Ex. Does the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converge?

Let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$ and $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$ For rational functions, use just highest degrees of num. and denom.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{2}{n^{1/2}}} = \lim_{n\to\infty} \frac{(2n^2+3n)n^{1/2}}{\sqrt{5+n^5}(2)} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2+0}{2\sqrt{0+1}} = 1 > 0$$

Since $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$ diverges (it is p-series with $p = \frac{1}{2} < 1$)

then by the Limit Comparison Test,

$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ diverges.