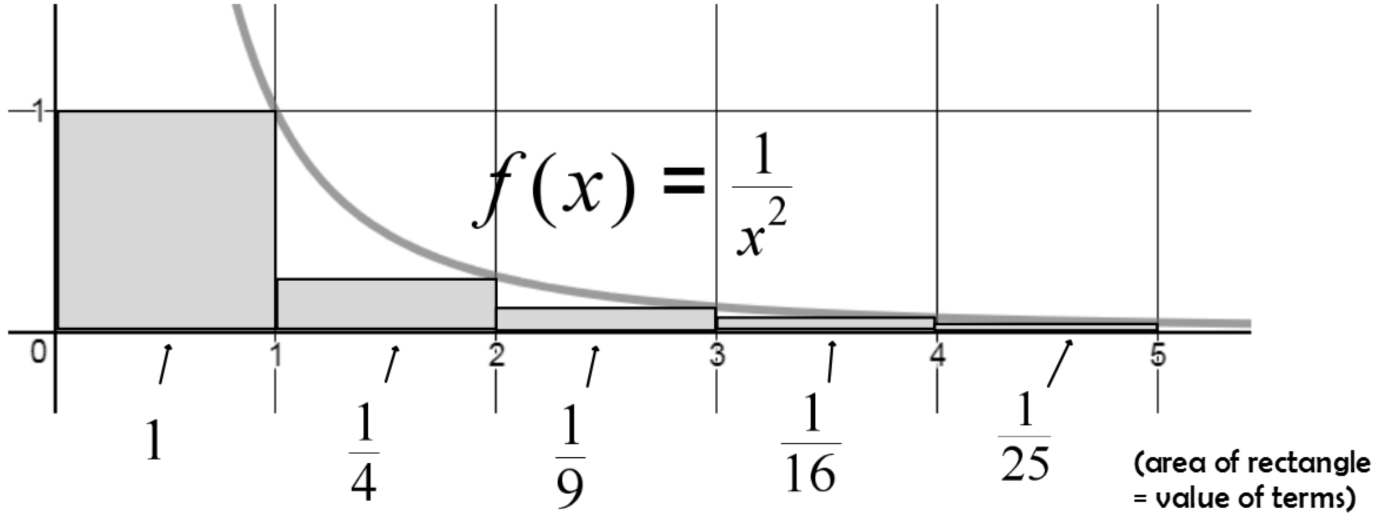
10.3 - The Integral Test

Last section we saw ways to come up with exact answer for a series (using a formula for geometric series)

Now we're just trying to determine if a series will converge, and attempt to estimate is sum.

Check out this series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$



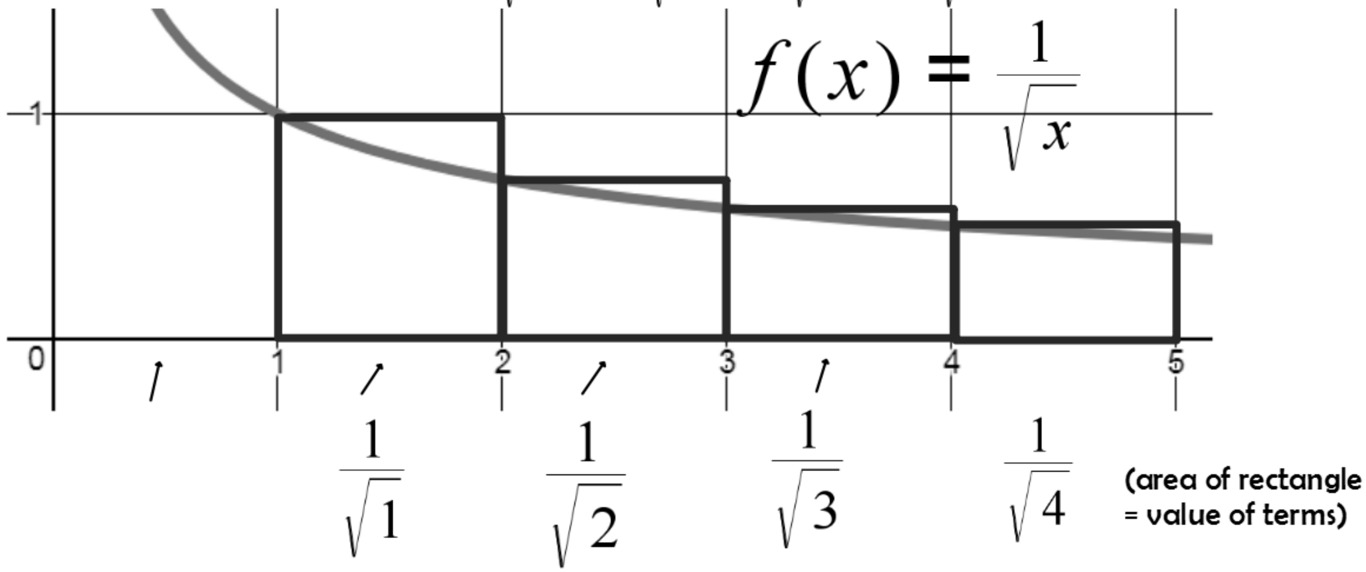
We can approximate the value of the series by noticing that the rectangles lie below the curve of the function.

$$1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + 1 = 2$$

Value of 1st term

While the value of the series is certainly below 2, we can tell here that the series will at least <u>converge</u>.

Check out this series: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$



In this case the rectangle are above the curve. Notice that for this function the rectangles are formed by left endpoints.

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty$$

Since the value of the series is greater than that of the integral (which is already divergent), then the series must be <u>divergant</u> as well.

The Integral Test

Let $a_n = f(n)$, and if f is a continuous, positive and decreasing function on $[1, \infty)$, then

- if $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- if $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{x(\ln x)^2}$ converges or diverges.

Since the function $f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive and decreasing on the interval [2, ∞), we can use the Integral Test.

it doesn't have to be 1

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \left[\frac{-1}{\ln x} \right]_2^b$$

$$= -\lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

The series converges.

P-series

The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if p > 1 and divergent if p ≤ 1.

Are the following series convergent or divergent?

$$\sum_{n=1}^{\infty}\frac{1}{n^{\pi}}$$ p = π , π > 1 convergent

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n^4}}$$ p = 4/5, 4/5 < 1 divergent