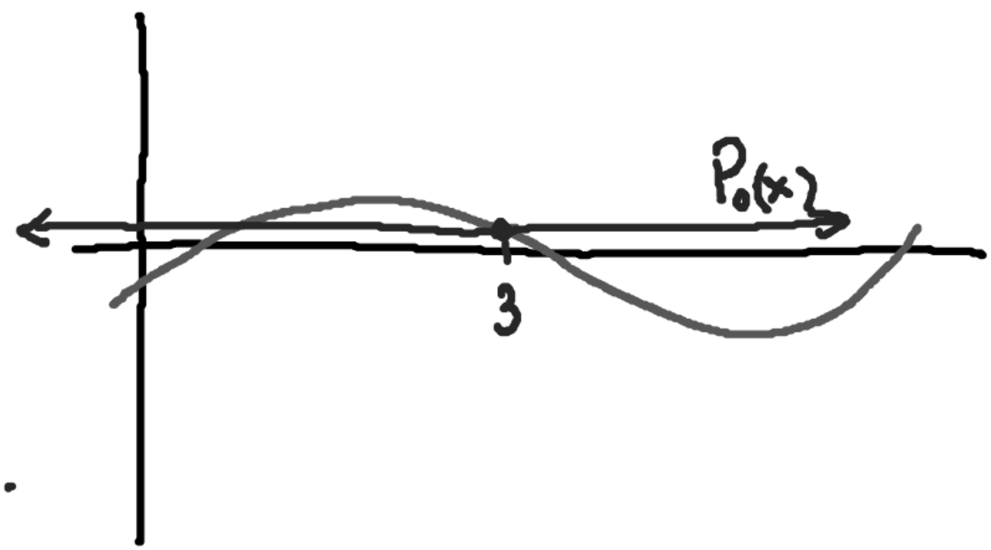
10.10 - Taylor Series

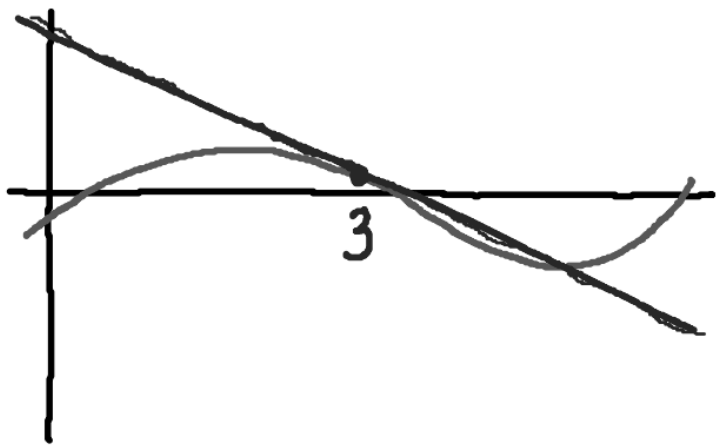
Say we had f(x). We want to approximate the function using a polynomial, and we'll use x = 3 as where we begin.



If we just used one term, our approximation would just be a single line.

We'll call our approximating function p(x), and all we can say now is that p(x) = f(3).

To increase our accuracy, let's make the derivatives match-up as well. So p'(x) will be the derivative of f(x) at x=3



$p(x) = f(3) + f'(3)(x-3)$

Make sure our functions still work:

$p(3) = f(3) + f'(3)(3-3)$
$p(3) = f(3)$

$p(x) = f(3) + f'(3)(x-3)$
$p(x) = f(3) + f'(3)x - f'(3)3$
$p'(x) = f'(3)$
$p'(3) = f'(3)$

Cool - so the polynomial we're making up, P(x), matches for the function value and the first derivative. So far though, our approximating function only is a tangent line, and doesn't do a great job of approximating f(x) at places away from x = 3.

Let's add another layer of complexity. Let's get the second derivatives to match. We want p"(x) = f "(x) at x = 3.

$$p(x) = f(3) + f'(3)(x-3) + \tfrac{1}{2}f''(3)(x-3)^2$$

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$$p(x) = f(3) + f'(x)(x-3) + \tfrac{1}{2}f''(x)(x-3)^2 + f'''(x)(x-3)^3$$
$$p'(x) = f'(3) + f''(3)(x-3)^1$$
$$p''(x) = f''(3)$$
$$p''(3) = f''(3)$$

3·2·1
3!

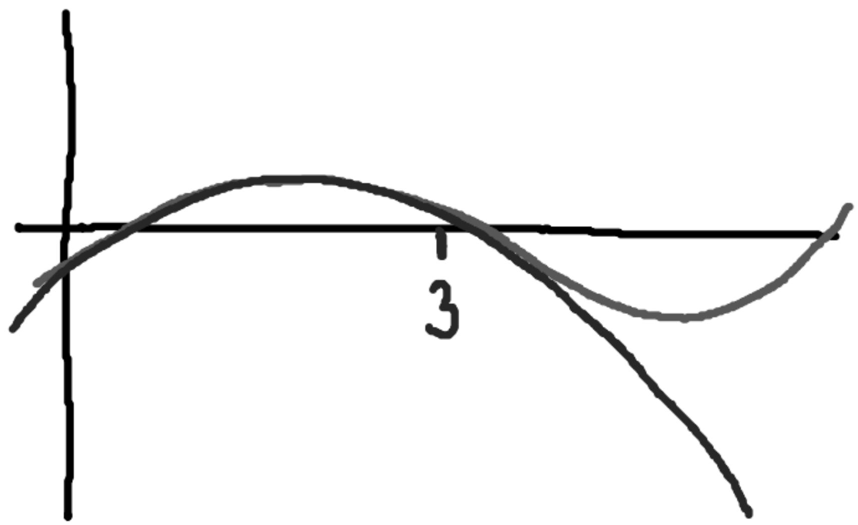
0! = 1

7(x−3)
7x−21

$_3C_3$

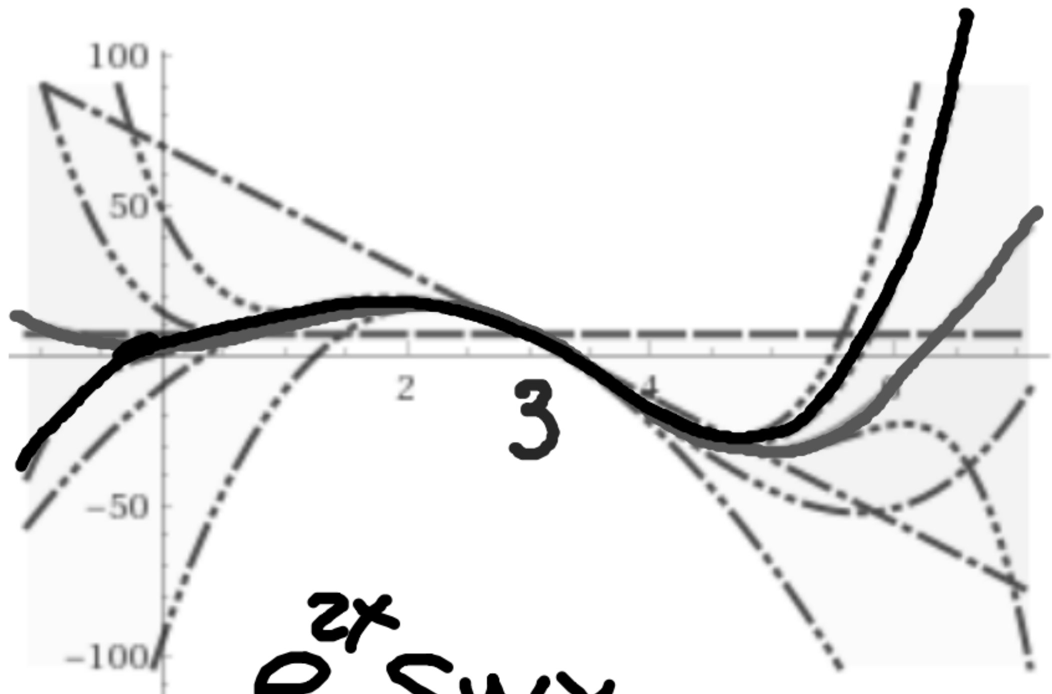
So now our function, first and second derivatives match at x = 3.

The second-degree function we made now looks something like this:



$$p(x) = f(3) + f'(3)(x-3) + \frac{1}{2}f''(3)(x-3)^2$$

As we keep adding more terms to our polynomial, we can better and better approximate f(x).



Σ $e^{2x}\sin x$

The polynomials we use to approximate functions are called Taylor polynomials. Learn this form:

$$P(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \frac{1}{3!}f'''(c)(x-c)^3 + \ldots$$
$$\ldots + \frac{1}{n!}f^n(c)(x-c)^n$$

c is where we're centering our function - where we're starting

Every coefficient will be $\boxed{\frac{1}{n!}f^n(c)}$

where n is both the degree of the term,
AND which derivative is being calculated.

Since we can't write out all the terms of a Taylor polynomial, a Taylor series can be used to represent f(x).

$$f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

For the special case where we're centered at 0, we call this series the Maclaurin series.

$$f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Find the Taylor series for f(x) centered at c = -2 for the function $f(x) = x - x^3$.

n	$f^{(n)}(x)$	$f^{(n)}(-2)$
0	$x - x^3$	6
1	$1 - 3x^2$	-11
2	$-6x$	12
3	-6	-6

$$f(x) = x - x^3 = \boxed{\sum_{n=0}^{3} \frac{f^{(n)}(-2)}{n!}(x+2)^n}$$

$$P(x) = \frac{6}{0!}(x+2)^0 + \frac{-11}{1!}(x+2)^1 + \frac{12}{2!}(x+2)^2 + \frac{-6}{3!}(x+2)^3$$

$$P(x) = 6 - 11(x+2) + 6(x+2)^2 - (x+2)^3$$

P(x) = f(x) , as f is a polynomial of finite terms. Finite series converge for all x, so R = ∞.

If a function has the following values at x = 0:

f(0) = -1, f '(0) = 6, f ''(0) = 2/3, f '''(0) = -6

Write the first four terms of the Maclaurin series

$$P(x) = -1 + 6x + \frac{x^2}{3} - x^3$$

What is the coefficient for the term containing $(x - 1)^3$ in the Taylor polynomial centered at $x = 1$ for the function $f(x)$?

$$f(x) = 3x^3 - x^2 - 4x + 7$$

3

The n^{th} derivative of g at x = 0 is given by

$$g^{(n)}(0) = (-1)^n\left(\frac{(n-1)!}{n+3}\right)$$

What is the coefficient for the term containing x^4 in the Maclaurin series of g?

$$\frac{1}{4!}g^{(4)}(0) = \frac{1}{4!}(-1)^4\left(\frac{(4-1)!}{4+3}\right)$$

$$\frac{1}{4!}g^{(4)}\left(0\right) = \frac{1}{4!}\left(\frac{3!}{7}\right)$$

$$\frac{1}{4!}g^{(4)}\left(0\right) = \frac{1}{4}\left(\frac{1}{7}\right) = \boxed{\frac{1}{28}}$$

↑ "formula" for the coefficient

What are the first two non-zero terms of the Maclaurin series for the function $f(x) = \cos(x + \pi)$?

$$-1 + \frac{x^2}{2}$$